Chapter 5: symmetries of fundamental field theories

In this chapter we explore how symmetries constrain the construction of an action for a theory of interest. We start by discussing the spacetime symmetries.

The Lorentz group

Minkowski space

Let us consider a photon moving with the speed of light c. It shall propagate for a distance $d\,x$ in an infinitesimal time dt, i.e.,

$$c^2 dt^2 - d\,x^2 = 0\,. \tag{1}$$

If we consider a transformation of space and time coordinates (t, x) to (t', x') the statement that the speed of light c is the same in the new coordinate system is equivalent to the statement that also

$$c^2 dt'^2 - d\,x'^2 = 0\,. \tag{2}$$

This property can now be expressed in a convenient mathematical representation by introducing vectors in a four-dimensional pseudo-Euclidean vector space with metric

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{3}$$

This vector space is called **Minkowski space**. The infinitesimal time dt and the corresponding vector $d\,x$ are combined to a four vector $(dx^\mu) = (c\,dt, d\,x)$ and the above equation can be written as

$$ds^2 = dx_\mu dx^\mu = g_{\mu\nu} dx^\mu dx^\nu = 0\,. \tag{4}$$

In this framework the transformations of coordinates that leave the speed of light invariant are just the isometries that leave the inner products of infinitesimal difference vectors invariant.

For convenience we adopt natural units in the remainder of this chapter and set $c = 1$ (as we previously also had set $\hbar = 1$).

Poincaré group

The group of isometries of the Minkowski space is the **Poincaré group** consisting of all transformations of the affine form

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + T^\mu\,, \tag{5}$$

with

$$dx'^\mu dx'_\mu = dx^\mu dx_\mu \tag{6}$$

where dx^μ is an infinitesimal difference vector in Minkowski spacetime, $\Lambda^\mu{}_\nu$ is a real four-by-four matrix and T^μ is a four vector describing translations. The above equation implies that

$$dx'^\mu dx'_\mu = g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta dx^\alpha dx^\beta = g_{\alpha\beta} dx^\alpha dx^\beta \tag{7}$$

or

$$g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}\,. \tag{8}$$

This defining condition can be written in matrix form as

$$\Lambda^T g \Lambda = g \tag{9}$$

and thus

$$\det \Lambda = \pm 1\,. \tag{10}$$

The subgroup defined by $T^\mu = 0$, i.e., the subgroup of all linear transformations, is the **Lorentz group** and its elements are called **Lorentz transformations**.

Restricted Lorentz group

Let us first consider Lorentz transformations that are continuously connected to the identity transformation $\Lambda = 1$. Lorentz transformations with this property live in a subgroup called the **restricted Lorentz group**. Since a continuous transformation cannot change the sign of $\det \Lambda$, restricted Lorentz transformations have $\det \Lambda = 1$.

A well-known subgroup of the restricted Lorentz group is the group of rotations with

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}, \tag{11}$$

where $R^T R = 1$, $\det R = 1$. Now by first rotating the spatial components appropriately we can restrict the remaining discussion to the two-dimensional subspace of vectors $(dx^\mu) = (dt, dx, 0, 0)$. The relevant Lorentz transformations are then of the form

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & 0 & 0 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

Thus for infinitesimal transformations $\Lambda = 1 + G$ the defining condition of $\det \Lambda = 1 + \mathrm{Tr}\, G = \pm 1$ yields $G^T g + gG = 0$, and therefore

$$\begin{aligned} 0 &= \begin{pmatrix} G^0{}_0 & G^1{}_0 \\ G^0{}_1 & G^1{}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} G^0{}_0 & G^0{}_1 \\ G^1{}_0 & G^1{}_1 \end{pmatrix} \\ &= \begin{pmatrix} G^0{}_0 & -G^1{}_0 \\ G^0{}_1 & -G^1{}_1 \end{pmatrix} + \begin{pmatrix} G^0{}_0 & G^0{}_1 \\ -G^1{}_0 & -G^1{}_1 \end{pmatrix}, \end{aligned} \tag{13}$$

or $G^0{}_0 = G^1{}_1 = 0$ and $G^0{}_1 = G^1{}_0$. A finite transformation is thus given by

$$\Lambda = \exp \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \tag{14}$$

with arbitrary $s \in \mathbb{R}$. Let us try to understand what the parameter s means. Consider an infinitesimal vector (dt, dx) that transforms to

$$\begin{pmatrix} dt' \\ dx' \end{pmatrix} = \Lambda \begin{pmatrix} dt \\ dx \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} dt \cosh s + dx \sinh s \\ dt \sinh s + dx \cosh s \end{pmatrix}. \tag{16}$$

Now we define a transformed velocity

$$v' = \frac{dx'}{dt'} = \frac{v \cosh s + \sinh s}{v \sinh s + \cosh s} \tag{17}$$

with $v = dx/dt$. If we have $v = 0$ in the untransformed system we have $v' = \tanh s$ in the transformed system. Therefore transformations of this type describe a change of coordinates to a frame of reference that moves with a constant velocity of $\tanh s$ relative to the original frame of reference. These are the **boosts** in the special theory of relativity with **rapidity** s.

Let us define $\beta = \tanh s$. Since $\cosh^2 s - \sinh^2 s = 1$, we can show that

$$\cosh s = \frac{1}{\sqrt{1 - \tanh^2 s}} = \frac{1}{\sqrt{1 - \beta^2}} = \gamma\,. \tag{18}$$

Therefore we can express the transformation also by the matrix

$$\Lambda(\beta) = \begin{pmatrix} \gamma(\beta) & \gamma(\beta)\beta \\ \gamma(\beta)\beta & \gamma(\beta) \end{pmatrix}. \tag{19}$$

Discrete Lorentz transformations

Consider the vector $(x^\mu) = (t, 0)$ which is invariant under rotations and transforms to

$$(x'^\mu) = \begin{pmatrix} t \cosh s \\ t \sinh s \end{pmatrix} \tag{20}$$

under a boost with rapidity s. Since $\cosh s > 0$, we conclude that the sign of x^0 is invariant under boosts and thus under the complete restricted Lorentz group.

Therefore, in order to obtain all possible Lorentz transformations, the discrete Lorentz transformation

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{21}$$

needs to be included in addition to restricted Lorentz transformations. This is the **time reversal** operator. Furthermore the **space inversion** or **parity** operator

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{22}$$

is also not a part of the restricted Lorentz group and needs to be included separately.

The quotient group of the Lorentz group and the restricted Lorentz group is the discrete group with elements

$$1, P, T, PT\,. \tag{23}$$

In other words, the Lorentz group can be partitioned in four disconnected parts defined by

$$\det \Lambda = \pm 1\,, \qquad \mathrm{Sign}\, \Lambda^0{}_0 = \pm 1\,. \tag{24}$$

We call transformations with $\det \Lambda = 1$ **proper** Lorentz transformations and transformations with $\mathrm{Sign}\, \Lambda^0{}_0 = 1$ **orthochronous** Lorentz transformations.

Generators of the restricted Lorentz group

Recall that infinitesimal restricted Lorentz transformations $\Lambda = 1 + G$ satisfy

$$G^T g + gG = 0\,. \tag{25}$$

We write G in block form

$$G = \begin{pmatrix} G_{00} & G_{01} \\ G_{10} & G_{11} \end{pmatrix}, \tag{26}$$

where G_{00} only acts on the temporal component, G_{11} only acts on the spatial components, and G_{01} and G_{10} mix spatial and temporal components. In this way the above equation can be expressed as

$$\begin{aligned} 0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1_3 \end{pmatrix} \begin{pmatrix} G_{00} & G_{01} \\ G_{10} & G_{11} \end{pmatrix} + \begin{pmatrix} G_{00} & G_{10}^T \\ G_{01}^T & G_{11}^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1_3 \end{pmatrix} \\ &= \begin{pmatrix} 2G_{00} & G_{01} - G_{10}^T \\ G_{01}^T - G_{10} & -G_{11} - G_{11}^T \end{pmatrix}, \end{aligned} \tag{27}$$

where 1_3 is the three-dimensional identity matrix. Therefore the defining conditions for generators of the restricted Lorentz group are

$$G_{01} = G_{10}^T\,, \qquad G_{00} = 0\,, \qquad G_{11}^T = -G_{11}\,. \tag{28}$$

This implies the following generators of the restricted Lorentz group.

The boosts are generated by

$$K_i = \begin{pmatrix} 0 & e_i^T \\ e_i & 0 \end{pmatrix} \tag{29}$$

with $(e_i)_j = \delta_{ij}$ and $i = 1, 2, 3$. They satisfy

$$\begin{aligned} [K_i, K_j] &= \begin{pmatrix} e_i^T e_j - e_j^T e_i & 0 \\ 0 & (e_i e_j^T - e_j e_i^T)_{ab} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (\delta_{ia}\delta_{jb} - \delta_{ja}\delta_{ib})_{ab} \end{pmatrix} \\ &= -\varepsilon_{ijk} J_k \end{aligned} \tag{30}$$

with J_k defined below.

The rotations are generated by

$$J_i = \begin{pmatrix} 0 & 0 \\ 0 & L_i \end{pmatrix} \tag{31}$$

with $(L_i)_{jk} = -\varepsilon_{ijk}$ and $i = 1, 2, 3$. They satisfy

$$\begin{aligned} [J_i, J_j] &= \begin{pmatrix} 0 & 0 \\ 0 & ([L_i, L_j])_{ab} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (\varepsilon_{ial}\varepsilon_{jlb} - \varepsilon_{jal}\varepsilon_{ilb})_{ab} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & (\delta_{ib}\delta_{aj} - \delta_{jb}\delta_{ia})_{ab} \end{pmatrix} = \varepsilon_{ijk} J_k\,. \end{aligned} \tag{32}$$

Hence boosts do not form a subgroup of the restricted Lorentz group, but rotations do.

Note that

$$[K_i, J_j] = \begin{pmatrix} 0 & e_i^T \\ e_i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & L_j \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & L_j \end{pmatrix} \begin{pmatrix} 0 & e_i^T \\ e_i & 0 \end{pmatrix} \tag{33}$$

$$= \begin{pmatrix} 0 & (e_i^T L_j)_a \\ (-L_j e_i)_a & 0 \end{pmatrix} \tag{34}$$

$$= \begin{pmatrix} 0 & \varepsilon_{ija} \\ \varepsilon_{ija} & 0 \end{pmatrix} = \varepsilon_{ijk} K_k \tag{35}$$

The Lie algebra of the restricted Lorentz group is therefore given by

$$[K_i, K_j] = -\varepsilon_{ijk} J_k\,, \qquad [J_i, J_j] = \varepsilon_{ijk} J_k\,, \qquad [K_i, J_j] = \varepsilon_{ijk} K_k\,. \tag{36}$$

A finite transformation is given by

$$\Lambda = \exp[\, s \cdot\, K + \,\phi \cdot\, J]\,, \tag{37}$$

where ϕ contains the angles of a rotation and s contains the rapidities of a boost.

A convenient representation of the generators is given by

$$S_i^{\pm} = \frac{1}{2}(\pm K_i + i J_i) \tag{38}$$

with $(S_i^{\pm})^\dagger = S_i^{\pm}$ and $i = 1, 2, 3$. We find

$$\begin{aligned} [S_i^a, S_j^b] &= (ab[K_i, K_j] + ib[J_i, K_j] + ia[K_i, J_j] - [J_i, J_j])/4 \\ &= i\varepsilon_{ijk}[i[(1 + ab)/4] J_k + [(a + b)/4] K_k] = \delta_{ab} i \varepsilon_{ijk} S_k^a\,. \end{aligned} \tag{39}$$

Therefore the group algebra factorizes in a direct product of two $\mathrm{SU}(2)$ algebras (this is of course not true in terms of groups). We can express J_i and K_i in terms of $S_i^{\pm}$ as

$$iJ_i = S_i^+ + S_i^-\,, \qquad K_i = S_i^+ - S_i^-\,. \tag{40}$$

Therefore $\Lambda = \exp[\, s \cdot\, K + \,\phi \cdot\, J]$ can be written as

$$\Lambda = \exp[s_i(S_i^+ - S_i^-) - i\phi_i(S_i^+ + S_i^-)] = \exp[-i x_i S_i^+] \exp[-i x_i^* S_i^-] \tag{41}$$

with $x_i = \phi_i + is_i$.

Translations in space and time

The Casimir operators of S^+ and S^- can now be used to classify the representations of the restricted Lorentz group. These Casimir operators are, however, no invariants of representations of the complete Poincaré group since they do not commute with all translations of space and time. In this section we show that the spin of a massive particle is, nevertheless, a well-defined quantity.

We extend the Minkowski space by a fifth dimension so that we can express a general transformation of the Poincaré group conveniently as

$$x' = \Gamma(\Lambda, T)x \tag{42}$$

with Lorentz transformation Λ, a four-dimensional translation vector (T_μ), $(x_\mu) = (x_0, x_1, x_2, x_3, 1)$, and

$$\Gamma(\Lambda, T) = \begin{pmatrix} \Lambda & (T_\mu) \\ 0 & 1 \end{pmatrix} \tag{43}$$

in block notation. The generators of translations in space and time P_μ are therefore given by the matrices

$$P_\mu = \begin{pmatrix} 0 & (\delta_{\mu\nu}) \\ 0 & 0 \end{pmatrix} \tag{44}$$

in block notation. A finite translation is given by

$$\Gamma(1, T) = \exp\left[\sum_{\mu=0}^{3} T_\mu P_\mu\right]. \tag{45}$$

We can now determine the algebra of the complete Poincaré group,

$$\begin{aligned} [P_\mu, P_\nu] &= 0\,, & [P_0, J_i] &= 0\,, & [P_0, K_i] &= -P_i\,, \\ [P_i, J_j] &= \varepsilon_{ijk}P_k\,, & [P_i, K_j] &= -\delta_{ij}P_0\,, & [K_i, K_j] &= -\varepsilon_{ijk}J_k\,, \\ [J_i, J_j] &= \varepsilon_{ijk}J_k\,, & [K_i, J_j] &= \varepsilon_{ijk}K_k\,. \end{aligned} \tag{46}$$

The Poincaré algebra has two Casimir operators. The first one is given by

$$C_1 = P_\mu P^\mu = P_0^2 - P_i^2\,. \tag{47}$$

Note that for the matrix P_μ given above this always vanishes, however, we can also consider different representations satisfying the same algebra with different C_1. We check explicitly that

$$[P_\mu, C_1] = 0\,, \tag{48}$$

$$\begin{aligned} [J_i, C_1] &= [J_i, P_0^2] - [J_i, P_j^2] = -[J_i, P_j]P_j - P_j[J_i, P_j] \\ &= 2\varepsilon_{ijk}P_kP_j = -2\varepsilon_{ijk}P_kP_j = 0\,, \end{aligned} \tag{49}$$

$$\begin{aligned} [K_i, C_1] &= [K_i, P_0]P_0 + P_0[K_i, P_0] - [K_i, P_j]P_j - P_j[K_i, P_j] \\ &= 2P_iP_0 - 2P_iP_0 = 0 \end{aligned} \tag{50}$$

for arbitrary i and μ. Let us pause at this point and ask what this means for a theory of a free particle with energy E and momentum p. In quantum mechanics the generator of the translations in space, P_i, measures the ith component of the momentum, and the generator of the translations in time, P_0, measures the energy. Therefore if we let C_1 act on a free particle state $|E,\ p\rangle$ with energy E and momentum p we find

$$C_1|E,\ p\rangle = (E^2 -\ p^2)|E,\ p\rangle = m^2|E,\ p\rangle\,, \tag{51}$$

where m is the mass of the particle. We can conclude that the mass of a particle is invariant under the Poincaré group and can be considered a well-defined property of a particle.

The second Casimir operator C_2 can be conveniently defined in terms of the **Pauli-Lubanski vector** W^μ with

$$W_0 = J_jP_j\,, \qquad W_i = P_0J_i - \varepsilon_{ijk}K_jP_k\,. \tag{52}$$

It is given by

$$C_2 = W^\mu W_\mu = (W_0)^2 - (W_i)^2\,. \tag{53}$$

In order to prove that C_2 is indeed a Casimir operator we first show that W_μ commutes with translations, i.e.,

$$[P_\mu, W_0] = [P_\mu, J_jP_j] = [P_\mu, J_j]P_j = (1 - \delta_{\mu 0})\varepsilon_{\mu jk}P_kP_j = 0\,, \tag{54}$$

$$[P_j, W_i] = P_0[P_j, J_i] - \varepsilon_{ilk}[P_j, K_l]P_k = P_0P_k(\varepsilon_{jik} + \varepsilon_{ijk}) = 0\,, \tag{55}$$

$$[P_0, W_i] = -\varepsilon_{ilk}[P_0, K_l]P_k = \varepsilon_{ilk}P_lP_k = 0\,. \tag{56}$$

Next we discuss the commutators of W_μ with boosts and calculate

$$\begin{aligned} [K_j, W_0] &= [K_j, J_iP_i] = [K_j, J_i]P_i + J_i[K_j, P_i] \\ &= \varepsilon_{jik}K_kP_i + J_jP_0 = W_j \end{aligned} \tag{57}$$

and

$$\begin{aligned} [K_j, W_i] &= [K_j, P_0J_i] - \varepsilon_{ilk}[K_j, K_lP_k] \\ &= P_0[K_j, J_i] + [K_j, P_0]J_i - \varepsilon_{ilk}K_l[K_j, P_k] - \varepsilon_{ilk}[K_j, K_l]P_k \\ &= \varepsilon_{jik}P_0K_k + P_jJ_i - \varepsilon_{ilj}K_lP_0 + \varepsilon_{ilk}\varepsilon_{jlr}J_rP_k \\ &= \varepsilon_{jik}[P_0, K_k] + P_jJ_i + (\delta_{ij}\delta_{kr} - \delta_{ir}\delta_{kj})J_rP_k \\ &= -\varepsilon_{jik}P_k + [P_j, J_i] + \delta_{ij}J_kP_k = -\varepsilon_{jik}P_k + \varepsilon_{jik}P_k + \delta_{ij}J_kP_k \\ &= \delta_{ij}W_0\,. \end{aligned} \tag{58}$$

We finally calculate the commutators of W_μ with rotations and find

$$\begin{aligned}
[J_j, W_0] &= [J_j, J_i P_i] = [J_j, J_i] P_i + J_i [J_j, P_i] = \varepsilon_{jik} J_k P_i - \varepsilon_{ijk} J_i P_k \\
&= \varepsilon_{jik} J_k P_i - \varepsilon_{kji} J_k P_i = 0\,, \qquad (59)\\
[J_j, W_i] &= [J_j, P_0 J_i] - \varepsilon_{ilk} [J_j, K_l P_k] \\
&= \varepsilon_{jik} P_0 J_k - \varepsilon_{ilk} [J_j, K_l] P_k - \varepsilon_{ilk} K_l [J_j, P_k] \\
&= \varepsilon_{jik} P_0 J_k + \varepsilon_{ilk} \varepsilon_{ljr} K_r P_k + \varepsilon_{ilk} \varepsilon_{kjr} K_l P_r \\
&= \varepsilon_{jik} P_0 J_k + (\varepsilon_{ilk} \varepsilon_{ljr} + \varepsilon_{ljk} \varepsilon_{irl}) K_r P_k \\
&= \varepsilon_{jik} P_0 J_k + (\delta_{kj} \delta_{ir} - \delta_{jr} \delta_{ik}) K_r P_k \\
&= \varepsilon_{jik} P_0 J_k + \varepsilon_{lji} \varepsilon_{lkr} K_r P_k \\
&= \varepsilon_{jik} (P_0 J_k + \varepsilon_{klr} K_r P_l) = \varepsilon_{jik} W_k\,. \qquad (60)
\end{aligned}$$

We observe that W_μ has the same commutation relations with the other parts of the algebra as P_μ, and therefore C_2 is also a Casimir operator.

For a massive particle we can calculate the action of C_2 in its rest frame, i.e.,

$$C_2 |m, 0\rangle = -m^2 J_i^2 |m, 0\rangle\,. \tag{61}$$

Therefore $|m, 0\rangle$ must also be an eigenstate of J_i^2 and the corresponding eigenvalues $s(s+1)$ correspond to the **spin** or intrinsic rotation of the point-like particle. In other words, massive particles can be classified according to their spin as defined by their behavior under the rotation group.

For a massless particle there is no rest frame and thus the situation is more complicated. It turns out that for massless particles the projection of the spin to the momentum,

$$\lambda = \ J \cdot \hat{P}\,, \tag{62}$$

is a well-defined property and assumes the role of the spin of massive particles. This property is called **helicity**.

For a detailed discussion of the representation theory of the complete Poincaré group, see, e.g., books of Weinberg or Ryder.

In the following discussion, we construct a Lagrangian of massive spin 0 and spin $1/2$ particles that are invariant under orthochronous Lorentz transformations.

Spinor representations

Note that the sub-sectors $+$ and $-$ of the restricted Lorentz group both transform identically under rotations with

$$\Lambda = \exp[-i\phi_i S_i^\pm]\,. \tag{63}$$

Since ϕ_i are the angles of a rotation in space and the S_i span the algebra of $\mathrm{SU}(2)$ the different representations of S correspond to different spin states. Possible representations of $S^+ \oplus S^-$ are

$$0 \oplus 0\,, \qquad \frac{1}{2} \oplus 0\,, \qquad 0 \oplus \frac{1}{2}\,, \qquad \frac{1}{2} \oplus \frac{1}{2}\,, \qquad \ldots\,. \tag{64}$$

The

$$0 \oplus 0$$

representation is the singlet representation, i.e., fields ϕ transforming in this representation do not change under a restricted Lorentz transformation. Such fields therefore correspond to spin 0 particles. In the next lecture, we complete the construction of a minimal Lagrangian for a spin 0 particle and extend the discussion to spin $1/2$ particles as well.

The Lagrangian of spin 0 fields

We first consider the $0 \oplus 0$ representation of $S^+ \oplus S^-$ for which fields ϕ transform as

$$\phi \to \phi' = \phi \tag{65}$$

under the restricted Lorentz group. Additionally, the derivatives transform as

$$\partial_\mu \to \partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu \tag{66}$$

$$\partial^\mu \to \partial^{\mu\prime} = \Lambda^\mu{}_\nu \partial^\nu \tag{67}$$

such that

$$\partial_\mu \partial^\mu \to \partial_\alpha \partial^\beta \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \partial_\alpha \partial^\beta g^\alpha{}_\beta = \partial_\mu \partial^\mu\,. \tag{68}$$

We also remind the reader that since $|\det \Lambda| = 1$, the integral measure d^4x is invariant under the restricted Lorentz group.

The action of chapter 4

$$S = \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \int d^4x \frac{1}{2} \phi(x)(-\partial_\mu \partial^\mu - m^2)\phi(x) \tag{69}$$

is therefore invariant under the restricted Lorentz group. The factor of $1/2$ and for that matter any factor of the kinetic term (derivative term) can be absorbed in a re-definition of the fields, over which we will integrate. The only relevant quantity here is the relative coefficient of the ϕ^2 mass term and the kinetic term.

Linear terms in the field are redundant

Consider shifting the fields by a constant, i.e., $\phi(x) \to \phi_0 + \phi(x)$ then

$$S \to -\int d^4x \frac{1}{2} \phi_0^2 m^2 - \int d^4x \frac{1}{2} \phi(x)(2m^2\phi_0) - \int d^4x \frac{1}{2} \phi_0 \partial_\mu \partial^\mu \phi(x) + \int d^4x \frac{1}{2} \phi(x)(-\partial_\mu \partial^\mu - m^2)\phi(x) \tag{70}$$

so this transformation generates a linear term in $\phi(x)$, a constant term, and a term proportional to

$$\int d^4x\partial_\mu\partial^\mu\phi(x)\,. \tag{71}$$

Such a term, however, vanishes as long as we assume that ϕ vanishes at sufficiently long distances or we consider a path integral in which the fields live on a torus. Therefore linear terms in ϕ in the action can be absorbed by a constant shift in fields in the path integral and are therefore redundant.

Action dimensionality and higher-dimensional contributions

Finally, we note that the action must have mass dimension 0. This can be quickly verified for the 1d free particle case

$$[S]=[\int dtp^2/2/m]=[t]+2[p]-[m]=-1+2-1=0\,. \tag{72}$$

Because of this, any additional terms in the action with higher mass dimensions must also be accompanied with additional dimensionful constant prefactors. We will investigate the role of such terms when we discuss the continuum limit of field theories.

Klein-Gordon equation

The equations of motion

$$\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}=-(m^2+\partial_\mu\partial^\mu)\phi=0\,. \tag{73}$$

yield the Klein-Gordon equations of relativistic scalar particles

$$(m^2+\partial_\mu\partial^\mu)\phi=0\,. \tag{74}$$

The Lagrangian of spin $1/2$ fields

We now try to construct a theory of spin $1/2$ particles such as electrons, neutrons or protons. To this end we first consider fields that transform in the $(1/2)\oplus 0$ representations of $S^+\oplus S^-$. We set

$$S_i^+=\frac{1}{2}\sigma_i\,,\qquad S_i^-=0 \tag{75}$$

with Pauli matrices σ_i and consider two-dimensional spinors ψ^+ which transform as

$$\psi'^+=\exp[(s_i-i\phi_i)\sigma_i/2]\psi^+ \tag{76}$$

under the restricted Lorentz group.

Let us try to construct a Lagrangian with fields ψ^+. Each term in the Lagrangian has to satisfy the following properties: (i) Due to relativity each term has to be a Lorentz scalar. (ii) The Lagrangian has to be real. (iii) Each term has to have mass dimension of 4 (the action has to be dimensionless).

The mass term

One may be tempted to write down a simple mass term of the form

$$\mathcal{L}_{\text{mass}}=m(\psi^+)^\dagger\psi^+\,. \tag{77}$$

Unfortunately, such a term does not satisfy (i) and is therefore not allowed in the Lagrangian. We discuss how a proper mass term can be constructed if we consider the representation $(1/2)\oplus(1/2)$ of $S^+\oplus S^-$ below. This is the mass term relevant for QCD. It is, however, instructive to consider another way to construct an invariant mass term that involves only $(\psi^+)^T$ and ψ^+, the **Majorana** mass term $(\psi^+)^T\sigma_2\psi^+$. First note that $\sigma_i^T=\sigma_i(-1)^{\delta_{i2}}$ with anticommutator $\{\sigma_i,\sigma_j\}=2\delta_{ij}1$, and therefore

$$\sigma_i^T\sigma_2\sigma_i=(-1)^{\delta_{i2}}\sigma_i\sigma_2\sigma_i=(-1)^{\delta_{i2}}(-\sigma_2+2\delta_{i2}\sigma_i)=-\sigma_2\,, \tag{78}$$

where no sum over i is implied. Thus $\sigma_i^T\sigma_2=-\sigma_2\sigma_i$, and for infinitesimal transformations with coordinates $x_i\ll 1$ we find

$$\begin{aligned}(\psi^+)^T\sigma_2\psi^+\to&(\psi^+)^T(1-ix_i\sigma_i^T/2)\sigma_2(1-ix_i\sigma_i/2)\psi^+\\&=(\psi^+)^T\sigma^2\psi^+-(i/2)x_i(\psi^+)^T(\sigma_i^T\sigma^2+\sigma^2\sigma_i)\psi^+\\&=(\psi^+)^T\sigma^2\psi^+\,.\end{aligned} \tag{79}$$

In order to make this term real we need to also include its complex conjugate. Since σ_2 is purely imaginary we write

$$\mathcal{L}_{\text{Majorana mass}}=im((\psi^+)^T\sigma^2\psi^+-(\psi^+)^\dagger\sigma^2(\psi^+)^*)\,. \tag{80}$$

Note that for a two-component field $(\psi^+)^T=(a,b)$ we find $(\psi^+)^T\sigma^2\psi^+=i(ba-ab)$. Therefore if we consider a and b to be ordinary numbers, the Majorana mass term would vanish identically. However, in a quantized theory we will later show that a and b anticommute since they correspond to fermions, and the Majorana mass term is nonzero.

The kinetic term

In this subsection we consider terms of the form

$$(\psi^+)^\dagger R\psi^+\,, \tag{81}$$

where R contains objects that transform non-trivially under the restricted Lorentz group. We use the first non-trivial ansatz including Lorentz vectors

$$R=M_\mu v^\mu\,, \tag{82}$$

where v^μ is a contravariant vector, M_μ is a matrix in the two-dimensional spin space and the sum over μ is implied. Note that M_μ is not a Lorentz vector. Therefore under Lorentz transformations we find

$$R' = M_\mu v'^\mu = M_\mu \Lambda^\mu{}_\nu v^\nu\,. \tag{83}$$

In order to construct an invariant term we need

$$(\psi^+)^\dagger R\psi^+ = (\psi^+)^\dagger \exp[(s_i + i\phi_i)\sigma_i/2]R' \exp[(s_i - i\phi_i)\sigma_i/2]\psi^+\,. \tag{84}$$

Let us first consider a infinitesimal boost in r direction, i.e., $\phi = 0$, $s_i = \delta_{ir}s$ with $s \ll 1$, and

$$v'^\mu = v^\mu + sK_r{}^\mu{}_\nu v^\nu\,. \tag{85}$$

Now Eq. (84) gives

$$\begin{aligned} M_\mu v^\mu &\stackrel{!}{=} [1 + s\sigma_r/2]M_\mu v'^\mu[1 + s\sigma_r/2] \\ &= [1 + s\sigma_r/2]M_\mu[v^\mu + sK_r{}^\mu{}_\nu v^\nu][1 + s\sigma_r/2] \\ &= M_\mu v^\mu + s(M_\mu K_r{}^\mu{}_\nu v^\nu + \sigma_r M_\mu v^\mu/2 + v^\mu M_\mu \sigma_r/2)\,. \end{aligned} \tag{86}$$

This has to hold for all v^μ so that we need

$$0 = M_\nu K_r{}^\nu{}_\mu + \{\sigma_r, M_\mu\}/2 = M_0\delta_{r\mu} + \delta_{\mu 0}M_r + \{\sigma_r, M_\mu\}/2\,. \tag{87}$$

Now this means that

$$M_r = -\{\sigma_r, M_0\}/2\,, \qquad M_0\delta_{ri} = -\{\sigma_r, M_i\}/2\,. \tag{88}$$

Next, we consider a rotation about the r axis, i.e., $s = 0$, $\phi_i = \delta_{ir}\phi$ with $\phi \ll 1$, and

$$v'^\mu = v^\mu + \phi J_r{}^\mu{}_\nu v^\nu\,. \tag{89}$$

Now Eq. (84) gives

$$\begin{aligned} M_\mu v^\mu &\stackrel{!}{=} M_\mu v'^\mu = [1 + i\phi\sigma_r/2]M_\mu[v^\mu + \phi J_r{}^\mu{}_\nu v^\nu][1 - i\phi\sigma_r/2] \\ &= M_\mu v^\mu + \phi(M_\nu J_r{}^\nu{}_\mu v^\mu + i[\sigma_r/2, M_\mu]v^\mu)\,. \end{aligned} \tag{90}$$

This has to hold for all v^μ so that we need

$$0 = M_\nu J_r{}^\nu{}_\mu + i[\sigma_r/2, M_\mu] = -\varepsilon_{\mu ri}(1 - \delta_{\mu 0})M_i + i[\sigma_r/2, M_\mu]\,, \tag{91}$$

and thus

$$[\sigma_r, M_\mu] = i2\varepsilon_{r\mu i}(1 - \delta_{\mu 0})M_i\,. \tag{92}$$

For $\mu = 0$ this means that $[\sigma_r, M_0] = 0$ for arbitrary r. This is only satisfied for

$$M_0 = c1\,. \tag{93}$$

For $\mu = j$ with $j = 1, 2, 3$ this means that

$$[\sigma_r, M_j] = i2\varepsilon_{rji}M_i\,. \tag{94}$$

We know that this is satisfied by the Pauli matrices

$$M_j = \sigma_j\,. \tag{95}$$

We determine c from Eq. (88) and $\{\sigma_r, \sigma_i\} = 21\delta_{ri}$ and find $c = -1$. It is easy to check that if we would have considered the sector $-$ instead of $+$ the solution would be $c = 1$. We define $(M_\mu) = (\sigma_\mu^+) = (-1, \sigma_1, \sigma_2, \sigma_3)$ and $(\sigma_+^\mu) = (-1, -\sigma_1, -\sigma_2, -\sigma_3)$ so that

$$(\psi^+)^\dagger\sigma_\mu^+ v^\mu\psi^+ = (\psi^+)^\dagger\sigma_\nu^+ g^{\mu\nu}v_\mu\psi^+ = (\psi^+)^\dagger\sigma_+^\mu v_\mu\psi^+ \tag{96}$$

is invariant under the restricted Lorentz group. While σ_+^μ does not transform as a Lorentz vector, we can conclude that

$$(\psi^+)^\dagger\sigma_+^\mu\psi^+ \tag{97}$$

does transform as a Lorentz vector. Note that the relevant matrices for the $-$ sector are $(\sigma_\mu^-) = (1, \sigma_1, \sigma_2, \sigma_3)$ and $(\sigma_-^\mu) = (1, -\sigma_1, -\sigma_2, -\sigma_3)$.

By substituting $v_\mu = \partial_\mu$ we can thus construct an invariant kinetic term that only involves $+$ fields.

Chirality

Let us consider all orthochronous Lorentz transformations, i.e, let us include the parity operator in addition to the restricted Lorentz transformations. The action of parity is defined by

$$\Lambda(s, \phi)P = P^2\Lambda(s, \phi)P = P\Lambda(-s, \phi) \tag{98}$$

due to $P^2 = 1$, $PK_iP = -K_i$, and $PJ_iP = J_i$. Equation (98) has to hold for all representations, and therefore the action $D(\Lambda)$ of Lorentz transformations Λ on ψ_+ yields

$$(D(P)\psi'_+) = D(P)D(\Lambda(s, \phi))\psi_+ = D(\Lambda(-s, \phi))(D(P)\psi_+) \tag{99}$$

with $\psi'_+ = D(\Lambda(s, \phi))\psi_+$. We observe that the field $D(P)\psi^+$ transforms according to the $1/2$ representation of S^-. Therefore if we want to construct a theory that is also invariant under parity, we need to include a spin $1/2$ representation of S^- as well. The twofold structure that emerges from the $(1/2) \oplus (1/2)$ representation of $S^+ \oplus S^-$ is called **chirality**.

We consider a spinor

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \tag{100}$$

where $\psi_\pm$ transform according to the $1/2$ representation of $S^\pm$. The action of parity shall be given by

$$D(P)\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{101}$$

in accordance with Eq. (99). We can write down a mass term

$$\mathcal{L}_{mass} = m\bar{\psi}\psi \tag{102}$$

with

$$\bar{\psi} = \begin{pmatrix} \psi_+^\dagger & \psi_-^\dagger \end{pmatrix} \tag{103}$$

that is invariant under orthochronous Lorentz transformations, see Eq. (41), repeated here:

$$\Lambda = \exp[-ix_i S_i^+] \exp[-ix_i^* S_i^-]\,. \tag{104}$$

We already know that

$$(\psi^+)^\dagger \sigma_+^\mu \partial_\mu \psi^+ \tag{105}$$

and

$$(\psi^-)^\dagger \sigma_-^\mu \partial_\mu \psi^- \tag{106}$$

are both invariant under the restricted Lorentz group. Under parity we have $\psi^+ \leftrightarrow \psi^-$ and $\partial_i \to -\partial_i$ for $i = 1, 2, 3$ so that

$$\sigma_+^\mu \partial_\mu \leftrightarrow -\sigma_-^\mu \partial_\mu\,. \tag{107}$$

Therefore we can construct a real and Lorentz invariant kinetic term

$$\begin{aligned} \mathcal{L}_{kinetic} &= i[(\psi^-)^\dagger \sigma_-^\mu \partial_\mu \psi^- - (\psi^+)^\dagger \sigma_+^\mu \partial_\mu \psi^+] \\ &= \bar{\psi} i\gamma^\mu \partial_\mu \psi \end{aligned} \tag{108}$$

with

$$\gamma^\mu = \begin{pmatrix} 0 & -\sigma_+^\mu \\ \sigma_-^\mu & 0 \end{pmatrix}. \tag{109}$$

The factor i is needed since ∂_μ is anti-Hermitian, i.e.,

$$\begin{aligned} \langle \psi' | \partial_x | \psi \rangle &= \int dx\; \psi'^*(x) \partial_x \psi(x) = -\int dx (\partial_x \psi'^*(x)) \psi(x) \\ &= -\langle \psi | \partial_\mu | \psi' \rangle^* \end{aligned} \tag{110}$$

for arbitrary fields ψ and ψ' with vanishing spacetime boundary contributions.

We write out the gamma matrices as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{111}$$

with $i = 1, 2, 3$ and note that

$$\bar{\psi} = \psi^\dagger \gamma^0\,. \tag{112}$$

The total Lagrangian of a noninteracting, massive spin $1/2$ particle of mass m is thus given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi\,. \tag{113}$$

It is apparent that this Lagrangian is also invariant under translations of space and time. The corresponding equation of motion is the **Dirac equation** of a free spin $1/2$ field

$$(i\gamma^\mu \partial_\mu - m)\psi = (iD - m)\psi = 0 \tag{114}$$

with **Dirac operator** $D = \gamma^\mu \partial_\mu$.

Note that we do not have to consider ψ_- and ψ_+ as independent fields. If we identify

$$\psi_- = i\sigma^2 \psi_+^* \tag{115}$$

it follows from $\sigma_2 \sigma_i^* \sigma_2 = -\sigma_i$, see Eq. (78), that under restricted Lorentz transformations

$$\begin{aligned} \psi_-' &= i\sigma^2 [\exp[(s_i - i\phi_i)\sigma_i/2]\psi_+]^* = \exp[(s_i + i\phi_i)\sigma^2 \sigma_i^* \sigma^2/2] i\sigma^2 \psi_+^* \\ &= \exp[(-s_i - i\phi_i)\sigma_i/2]\psi_-\,, \end{aligned} \tag{116}$$

in accordance with Eq. (99). The mass terms then become Majorana mass terms, and it can be shown that the fields ψ_+ become their own antiparticles. This, however, implies that they are not allowed to carry a nonzero charge and therefore excludes this scenario for the quarks of QCD.

Gamma matrices and Lorentz structure

Before we continue with the discussion of gauge symmetries a few notes about the algebra of gamma matrices are in order. The gamma matrices satisfy the **Clifford-algebra** relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\,. \tag{117}$$

The parity operator can be written in terms of γ^0 as

$$D(P)\psi = \gamma^0 \psi\,. \tag{118}$$

Furthermore, it is convenient to define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \tag{119}$$

which allows to project on the $-$ and $+$ sectors by

$$P_\pm = \frac{1 \pm \gamma^5}{2}\,, \tag{120}$$

where 1 is the identity matrix in the respective space. The matrix γ^5 anticommutes with all other gamma matrices,

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{121}$$

with $\mu = 0, 1, 2, 3$.

Note that the gamma matrices can be used to construct field bilinears that transform in a well-defined way under the orthochronous Lorentz group. Under restricted Lorentz transformations Λ we find

$$\begin{aligned} v^\mu = \bar\psi\gamma^\mu\psi \quad &\to \quad \Lambda^\mu{}_\nu v^\nu\,, \\ a^\mu = \bar\psi\gamma^\mu\gamma^5\psi \quad &\to \quad \Lambda^\mu{}_\nu a^\nu\,, \\ s = \bar\psi\psi \quad &\to \quad s\,, \\ p = \bar\psi\gamma^5\psi \quad &\to \quad p\,, \end{aligned} \tag{122}$$

see Eq. (97). The action of parity P on v, a, s and p is given by

$$\begin{aligned} v^\mu \quad &\to \quad -v^\mu + 2g^{0\mu}v^0\,, \\ a^\mu \quad &\to \quad a^\mu - 2g^{0\mu}a^0\,, \\ s \quad &\to \quad s\,, \\ p \quad &\to \quad -p\,. \end{aligned} \tag{123}$$

Therefore v^μ transforms as a vector, a^μ transforms as an axial vector, s transforms as a scalar and p transforms as a pseudoscalar.

Gauge symmetry

In the last section we have constructed a relativistically invariant Lagrangian of a massive spin $1/2$ field. Up to now the particles represented by the field do not interact with each other. In the following we add a local internal symmetry (or **gauge symmetry**) to the Lagrangian and show that such a modification introduces an interaction between the spin $1/2$ particles that is mediated by massless spin 1 particles.

Internal symmetries

Consider the Lagrangian of Eq. (113), i.e.,

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi \tag{113}$$

with fields ψ in spinor space. The operation of the matrices γ^μ on ψ is given by the matrix-vector multiplication in this space. The most trivial way to add an additional symmetry S_i is to choose a new symmetry group S that is a direct product of the Poincaré symmetry group S_p and S_i,

$$S = S_p \otimes S_i\,. \tag{124}$$

In such a modification we call S_i an **internal symmetry** of the Lagrangian. The fields ψ must transform in representations of the bigger symmetry group S and therefore live in a product space of the spinor space and the vector space of the internal symmetry.

Local symmetries

Let us choose S_i to consist of spacetime-dependent transformations of $\psi(x)$ with infinitesimal transformations $G(x)$ defined by

$$\psi(x) \to \psi(x) + iG(x)\psi(x)\,, \tag{125}$$

where the action of $G(x)$ on $\psi(x)$ is the matrix-vector multiplication in the internal symmetry space. We ignore terms of order G^2 throughout the remainder of this section. The mass term of Eq. (113),

$$\mathcal{L}_{\text{mass}} = m\bar\psi\psi\,, \tag{126}$$

is symmetric under Eq. (125) if

$$G(x)^\dagger = G(x)\,, \tag{127}$$

i.e., if $G(x)$ generates unitary transformations. The kinetic term

$$\mathcal{L}_{\text{kinetic}} = \bar\psi(i\gamma^\mu\partial_\mu)\psi\,, \tag{128}$$

however, transforms to

$$\mathcal{L}'_{\text{kinetic}} = \mathcal{L}_{\text{kinetic}} - \bar\psi(\partial_\mu G(x))\gamma^\mu\psi \tag{129}$$

under unitary transformations. An invariant term can only be constructed if we replace

$$\partial_\mu \to D_\mu \tag{130}$$

with

$$D_\mu \to [1 + iG(x)]D_\mu[1 - iG(x)^\dagger] \tag{131}$$

under Eq. (125), where 1 is the identity matrix. We call D_μ a **covariant derivative**. The covariant derivative has to generate a kinetic term for the spin $1/2$ fields, and therefore we use the ansatz

$$D_\mu = \partial_\mu + iA_\mu\,, \tag{132}$$

where A_μ has to transform under S_i in a way that satisfies Eq. (131). Note that A_μ can act non-trivially on the internal symmetry space. Since ∂_μ is an anti-Hermitian operator, we require $A = A^\dagger$ so that the Lagrangian is real. In accordance with Eq. (131) we request that the transformed A'_μ satisfies

$$\begin{aligned}\partial_\mu + iA'_\mu &= [1+iG(x)](\partial_\mu + iA_\mu)[1-iG(x)^\dagger] \\ &= \partial_\mu + iA_\mu - i(\partial_\mu G(x)) - [G(x), A_\mu]\,,\end{aligned} \tag{133}$$

where we used the Hermiticity of $G(x)$. Thus we can construct an invariant kinetic term if A_μ transforms as

$$A'_\mu = A_\mu - (\partial_\mu G(x)) + i[G(x), A_\mu]\,. \tag{134}$$

We conclude that we can construct a Lagrangian

$$\mathcal{L} = \bar\psi(i\gamma^\mu D_\mu - m)\psi = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi - \bar\psi\gamma^\mu A_\mu\psi \tag{135}$$

that is invariant under the symmetry group S with internal symmetry S_i defined by the infinitesimal transformation

$$\begin{aligned}\psi(x) &\to \psi(x) + iG(x)\psi(x)\,, \\ A_\mu &\to A_\mu - (\partial_\mu G(x)) + i[G(x), A_\mu]\,.\end{aligned} \tag{136}$$

Note that the A_μ also transform in the fundamental representation of the restricted Lorentz symmetry group,

$$A_\mu \to {\Lambda_\mu}^\nu A_\nu \tag{137}$$

under Lorentz transformation Λ. The spin operator of the fundamental representation of the restricted Lorentz group is given by $S_j = iJ_j$ with $S^2 = s(s+1)$ and $s=1$. Therefore we have introduced fields A_μ of spin 1 that interact with the spin $1/2$ fields due to the term

$$\mathcal{L}_{\text{interaction}} = -\bar\psi\gamma^\mu A_\mu\psi \tag{138}$$

in the Lagrangian. Since the Lagrangian has to be invariant under Eq. (???) the fields A_μ are not allowed to have a quadratic mass term and must therefore correspond to massless particles. They can, however, have a kinetic term that allows them to propagate in spacetime. To second order in $\partial_\mu A_\nu$ the only term that is invariant under S_i and S_p is proportional to

$$\mathcal{L}_{\text{YM}} \propto \mathrm{Tr}\,[D_\mu, D_\nu][D^\mu, D^\nu]\,, \tag{139}$$

where the trace Tr acts on the internal symmetry space. This is the **Yang-Mills** term. The invariance under S_i is due to the covariance of

$$[D_\mu, D_\nu] \to [1+iG(x)][D_\mu, D_\nu][1-iG(x)^\dagger] \tag{140}$$

under Eq. (???). We define the **field-strength tensor**

$$F_{\mu\nu} = -i[D_\mu, D_\nu] = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) + i[A_\mu, A_\nu] \tag{141}$$

and express the total Lagrangian conveniently as

$$\mathcal{L} = \bar\psi(i\gamma^\mu D_\mu - m)\psi + \frac{1}{2}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}\,. \tag{142}$$

Note that if the local symmetry group is not abelian, the term $[A_\mu, A_\nu]$ introduces a self-interaction between the massless spin 1 particles.

If we choose the first non-trivial unitary symmetry group $\mathrm{U}(1)$, we recover the theory of electrodynamics coupled to a spin $1/2$ field. The photons are now given by the spin 1 fields A_μ. The equations of motion of the fields A_μ can readily be identified with Maxwell's equations of electrodynamics.

The Lagrangian of QCD

The internal symmetry group of QCD is given by $\mathrm{SU}(3)$. If we choose A_μ to live in the group algebra of $\mathrm{SU}(3)$, we can write

$$A_\mu = A^a_\mu\lambda_a\,, \tag{143}$$

where the matrices λ_a ($a = 1,\ldots,8$) span the algebra of $\mathrm{SU}(3)$. The eight fields A^a_μ now correspond to eight independent gluons. The quarks live in the internal symmetry space of $\mathrm{SU}(3)$. Its fundamental representation is three-dimensional and therefore there are three different quark fields, or three different **colors** of quarks (The name quantum chromodynamics is due to this interpretation of the three different quark fields as different colors of quarks.). The bound states of quarks and anti-quarks, called **hadrons**, must transform as singlets of S_i and are therefore **color neutral**. There are two types of hadrons: **mesons** and **baryons**. Mesons, such as the pion, are bosonic hadrons that consist of a quark and an anti-quark. Baryons, such as the proton or neutron, are fermionic hadrons that consist of three quarks.

Note that since $\mathrm{SU}(3)$ is non-abelian, gluons are self-interacting. This property can be shown to lead to the **asymptotic freedom** of QCD, i.e., for high energies the strength of the interaction becomes weaker.

We rescale the fields $A_\mu \to gA_\mu$ and change the prefactor of the kinetic term of gluons so that we can adjust the strength of the interaction of quarks and gluons explicitly. The Lagrangian of a quark coupled to the gluons then reads

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} - gA^a_\mu\bar\psi\gamma^\mu\lambda_a\psi \tag{144}$$

with

$$F_{\mu\nu} = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) + ig[A_\mu, A_\nu]\,. \tag{145}$$

It turns out that in nature there are more than one kind of quarks which differ by their mass and electromagnetic charge. One currently has experimental evidence for 6 different types of quarks, called different quark **flavors**, of which three have a fractional electromagnetic charge of $+2/3$ and three have a fractional electromagnetic charge of $-1/3$. Two quarks are very light and thus play an important role in the low-energy physics of QCD discussed in the remainder of this thesis. They are called **up** and **down** quarks (corresponding to their respective fractional electromagnetic charges $+2/3$ and $-1/3$). The next heavier quark is called **strange** quark and has a fractional electromagnetic charge of $-1/3$. Their masses are related approximately by

$$\frac{m_s}{m_d} \approx 20\,, \qquad \frac{m_u}{m_d} \approx \frac{1}{2}\,, \tag{146}$$

where m_u, m_d, m_s are the masses of up, down and strange quark. Note that these relations are only order-of-magnitude estimates. The total Lagrangian of QCD thus reads

$$\begin{aligned} \mathcal{L}_{\mathrm{QCD}} = \sum_{f=1}^{6} \bar{\psi}_f (i\gamma^\mu \partial_\mu - m_f)\psi_f + \frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \\ - g\sum_{f=1}^{6} A_\mu^a \bar{\psi}_f \gamma^\mu \lambda_a \psi_f\,. \end{aligned} \tag{147}$$

We observe that, depending on the quark masses, the total Lagrangian has an additional symmetry in flavor space. This symmetry will force some of the hadrons to be particularly light. More on this in a later chapter.