Universität Regensburg, Institut für Theoretische Physik Winter 2022/23
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Exercise: Quantum Computing
Problem set 11 (to be discussed in week of January 23rd, 2023)

Problem 1 Normalizer and centralizer

In the lecture, we defined the normalizer

$$N(S) \equiv \{E \in G_n | EgE^\dagger \in S, \forall g \in S\}\,. \tag{1}$$

Show that $Z(S) = N(S)$ if S does not contain -$\mathbb{1}$ with **centralizer** of S in G_n defined as

$$Z(S) \equiv \{E \in G_n | Eg = gE, \forall g \in S\}\,. \tag{2}$$

Hint: first show that $Z(S) \subseteq N(S)$ by showing every element of $Z(S)$ is in $N(S)$. Therefore if $N(S) \neq Z(S)$, there is a $E \in N(S)$ with $E \notin Z(S)$. Show that for such an E, we have $EgE^\dagger = -g \in S$ for a $g \in S$. Then since S is a group, $g^{-1} \in S$ and therefore $-gg^{-1} = -\mathbb{1} \in S$.

Problem 2 Stabilizer code [5,1]

Consider the [5,1] stabilizer code defined by the generators

$$K_1 = X_1 Z_2 Z_3 X_4\,, \qquad K_2 = X_2 Z_3 Z_4 X_5\,, \tag{3}$$
$$K_3 = X_1 X_3 Z_4 Z_5\,, \qquad K_4 = Z_1 X_2 X_4 Z_5 \tag{4}$$

and

$$\overline{Z} = \prod_{i=1}^{5} Z_i\,. \tag{5}$$

a) Construct the logical states $|0_L\rangle$ and $|1_L\rangle$ using the method described in the lecture of July 9.

b) Using the result of problem 1, show that this code corrects against X_l, Z_l, and $X_l Z_l$ errors for $l \in \{1, 2, 3, 4, 5\}$. It therefore corrects for arbitrary one-qubit errors. You may restrict yourself to show this only for the error set $\{\mathbb{1}, X_1, Z_1, X_1 Z_1\}$.

   Hint: from problem 1 it suffices to show that for any two errors E_i, E_j the product $E_i^\dagger E_j$ is either a product of generators or does not commute with at least one generator.

c) Using the construction of the lecture on July 9, create a circuit that corrects for a single such error. (Use 4 ancilla bits to store the syndromes β_l and apply the corresponding correction steps depending on the ancilla values.)

d) Implement this circuit in sqc (optional).