Universität Regensburg, Institut für Theoretische Physik
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Exercise: Quantum Computing
Problem set 9 (to be discussed in week of January 9, 2023)

Problem 1 One-qubit depolarization channel

First show that for a density matrix ρ, we have

$$\frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{2} = \mathbb{1}\,. \tag{1}$$

Using this identity, derive the operator sum representation of the depolarization channel

$$\mathcal{E}(\rho) = \frac{1-p}{2}\mathbb{1} + p\rho \tag{2}$$

with $0 \leq p \leq 1$, i.e., find matrices E_k for

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger\,. \tag{3}$$

Problem 2 Fidelity of one-qubit depolarziation channel

Let $\sigma = \mathcal{E}(\rho)$ for the one-qubit depolarization channel. Calculate the fidelity

$$F(\rho,\sigma) \equiv \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right]\,. \tag{4}$$

for a pure initial state $\rho = |\Psi\rangle\langle\Psi|$. You should find that the fidelity is independent of the state ρ.

Problem 3 Fidelity of phase decoherence channel

Let $\sigma = \mathcal{E}(\rho)$ for the one-qubit phase decoherence channel defined through Kraus operators

$$E_0 = \sqrt{\alpha}\mathbb{1}\,, \qquad E_1 = \sqrt{1-\alpha}Z \tag{5}$$

with $\alpha \in [0,1]$. Calculate the fidelity $F(\rho,\sigma)$ for a pure initial state $\rho = |\Psi\rangle\langle\Psi|$. In this case the fidelity will depend on $|\Psi\rangle$ such that it is useful to define

$$F_{\min} = \min_{|\Psi\rangle} F \tag{6}$$

as the fidelity minimized over all possible states $|\Psi\rangle$. What is $F_{\min}$?