Universität Regensburg, Institut für Theoretische Physik
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Exercise: Quantum Computing

Problem set 8 (to be discussed in week of December 19, 2022)

Problem 1 Operator sum representation of CNOT

The first entangled state that we discussed in the lecture was the Bell state that we prepared by a Hadamard followed by a CNOT gate on a two-qubit system. We will now consider a two-qubit system with least-significant qubit (0) being the principal system and the most-significant qubit (1) being the environment and describe the entangling CNOT in the operator sum representation.

Consider a system that originally is in the unentangled state $\rho \equiv \rho_{\mathrm{P}} \otimes \rho_{\mathrm{E}}$ with density matrices for the principal system ρ_{P} and for the environment ρ_{E}. Without loss of generality, you may assume $\rho_{\mathrm{E}} = |e_0\rangle\langle e_0|$, i.e., the environment starts in a pure state. (If it were not, you could use the purification technique described in the lecture.)

Let $\mathrm{CNOT}(i,j)$ be the CNOT gate with control bit i and target bit j and

$$\mathcal{E}_U(\rho_{\mathrm{P}}) \equiv \mathrm{Tr}_{\mathrm{E}}\left[U\rho U^\dagger\right] . \tag{1}$$

Find the operator sum representation, i.e., find operators E_k for which

$$\mathcal{E}_U(\rho_{\mathrm{P}}) = \sum_k E_k \rho_{\mathrm{P}} E_k^\dagger \tag{2}$$

for $U = \mathrm{CNOT}(0,1)$ as well as $U = \mathrm{CNOT}(1,0)$. Give explicit matrix representations of the E_k. It may be helpful to first write out the matrix representation of $U = U_{ab;cd}\,|ab\rangle\langle cd|$ with sum over repeated indices implied.

Problem 2 Generalized amplitude damping

Consider the generalized amplitude damping channel described by Kraus operators E_k in

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \tag{3}$$

with

$$E_0 = \sqrt{p}\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{pmatrix}, \qquad E_1 = \sqrt{p}\begin{pmatrix}0 & \sqrt{\gamma}\\ 0 & 0\end{pmatrix}, \tag{4}$$

$$E_2 = \sqrt{1-p}X\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{pmatrix}X\,, \qquad E_3 = \sqrt{1-p}X\begin{pmatrix}0 & \sqrt{\gamma}\\ 0 & 0\end{pmatrix}X\,, \tag{5}$$

NOT matrix X, and $\gamma \equiv 1 - e^{-t/T_1}$, and $p \in [0,1]$. This channel can describe the energy relaxation process with relaxation time T_1 at finite temperature T:

a) We can define a discrete time-series of density matrices ρ_n defined by

$$\rho_{n+1} = \mathcal{E}(\rho_n) \tag{6}$$

given a starting matrix ρ_0. One can show that at sufficiently large n for a general ρ_0, the system stabilizes to

$$\rho_\infty = \mathcal{E}(\rho_\infty)\,. \tag{7}$$

Find ρ_∞.

b) At finite temperature we expect a Boltzmann distribution $p_n = e^{-E_n/(k_B T)}/\mathcal{Z}$ with $\mathcal{Z} = \sum_n e^{-E_n/(k_B T)}$ with energies E_n of states n and Boltzmann constant k_B. In our case we have two states $|0\rangle$ and $|1\rangle$ with energies E_0 and E_1. What temperature does ρ_∞ correspond to?