Universität Regensburg, Institut für Theoretische Physik
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Exercise: Quantum Computing
Problem set 7 (to be discussed in week of December 12, 2022)

Problem 1 Stochastic energy fluctuations and decoherence times

In the lecture, we have discussed the Ramsey decoherence experiment for which the probability to measure a zero bit is given by

$$P_0^{\mathrm{R}}(t) = \frac{1}{2} + \frac{1}{2}\cos\left(\int_0^t d\tau \Delta E(\tau)\right) \tag{1}$$

and the Hahn echo decoherence experiment for which this probability is

$$P_0^{\mathrm{H}}(t) = \frac{1}{2} + \frac{1}{2}\cos\left(\int_0^{t/2} d\tau \Delta E(\tau) - \int_{t/2}^t d\tau \Delta E(\tau)\right) . \tag{2}$$

In both cases, $\Delta E(\tau)$ shall denote the precise energy splitting between the $|0\rangle$ and $|1\rangle$ eigenstates of the system for time τ. Fluctuations in this quantity introduce decoherence in the qubit. In this problem, we derive the quoted real probability distribution that defined the decoherence times T_2 and T_2^* from a simple model.
Consider the ansatz for the time-dependent energy splitting

$$\Delta E(\tau) \equiv \Delta \overline{E} + \eta(\tau) \tag{3}$$

with real positive $\Delta \overline{E}$ and a noise source η that we set to be constant for small time intervals Δt. Concretely, we define

$$\eta(i\Delta t + \Delta\tau) \equiv \frac{\eta_i}{\Delta t} \tag{4}$$

with integer i and $0 \leq \Delta\tau < \Delta t$. So for $0 \leq \tau < \Delta t$ we have $\eta(\tau) = \eta_0$, for $\Delta t \leq \tau < 2\Delta t$ we have $\eta(\tau) = \eta_1$, and so forth.
To complete the model, we demand that η_i and η_j are statistically independent for $i \neq j$ and that we have the expectation values

$$\langle \eta_i^{2n+1} \rangle_{\eta_i} = 0 \,, \qquad \langle \eta_i^2 \rangle_{\eta_i} = \frac{2\Delta t}{T} \,, \qquad \langle \eta_i^{2n+2} \rangle_{\eta_i} = O(\Delta t)^2 \tag{5}$$

for all i with $n \in \{0, 1, \ldots\}$ and positive real T. In practice, η_i could, e.g., be drawn from a Gaussian distribution with center 0 and variance $2\Delta t/T$, however, the actual distribution is not important as long as Eq. (5) is satisfied.
Using the following steps, show that in this model $T_2 = T_2^* = T$.

a) Show that $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ for real a and b. You may use the relation of the trigonometric functions to the complex exponential function to do this.

b) Show that

$$\phi_i^{\mathrm{R}} \equiv \int_0^{i\Delta t} d\tau \Delta E(\tau) = i\Delta t \Delta \overline{E} + \sum_{j=0}^{i-1} \eta_j \tag{6}$$

and

$$\phi_i^{\mathrm{H}} \equiv \int_0^{i\Delta t} d\tau \Delta E(\tau) - \int_{i\Delta t}^{2i\Delta t} d\tau \Delta E(\tau) = \sum_{j=0}^{i-1} \eta_j - \sum_{j=i}^{2i-1} \eta_j \, . \tag{7}$$

c) Show that

$$\langle \cos(\phi + \eta_i) \rangle_{\eta_i} = \cos(\phi) \left(1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right) \tag{8}$$

where ϕ is a number that does not depend on η_i. Hint: consider the Taylor expansion of $\sin(a)$ for a real a.

d) Show that

$$\langle P_0^{\mathrm{R}}(t) \rangle_{\eta_0, \eta_1, \ldots} = \frac{1}{2} + \frac{1}{2} \cos(t \Delta \overline{E}) \left(1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)^{t/\Delta t} \tag{9}$$

and

$$\langle P_0^{\mathrm{H}}(2t) \rangle_{\eta_0, \eta_1, \ldots} = \frac{1}{2} + \frac{1}{2} \left(1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)^{2t/\Delta t} \tag{10}$$

for all t that can be written as $t = i\Delta t$ for integer $i \geq 0$.

e) Show that

$$\lim_{\Delta t \to 0} \left(1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)^{t/\Delta t} = e^{-t/T} \, . \tag{11}$$

It may be helpful to consider the logarithm of both sides.

f) Show that by taking the limit of $\Delta t \to 0$, we find

$$\langle P_0^{\mathrm{R}}(t) \rangle_{\eta_0, \eta_1, \ldots} = \frac{1}{2} + \frac{1}{2} \cos(t \Delta \overline{E}) e^{-t/T} \tag{12}$$

and

$$\langle P_0^{\mathrm{H}}(t) \rangle_{\eta_0, \eta_1, \ldots} = \frac{1}{2} + \frac{1}{2} e^{-t/T} \tag{13}$$

for any real $t \geq 0$ and therefore $T_2 = T_2^* = T$.