Universität Regensburg, Institut für Theoretische Physik
Prof. Dr. Christoph Lehner, Andreas Hackl Winter 2022/23

Exercise: Quantum Computing
Problem set 4 (to be discussed in week of November 21, 2022)

Problem 1 Phase estimation success rate

In the lecture, we discussed that the probability of measuring a

$$\tilde{\phi} = \frac{x}{2^N} \tag{1}$$

in the phase estimation algorithm given a true eigenvalue $\phi \in [0, 1[$ that satisfies

$$|\phi - \tilde{\phi}| \leq 2^{-n} \tag{2}$$

is at least $1 - \varepsilon$ for

$$N = n + \log_2 \left(2 + \frac{1}{2\varepsilon}\right) \tag{3}$$

qubits. Derive this limit through the following steps.

a) Show

$$\sum_{i=0}^{m-1} x^i = \frac{x^m - 1}{x - 1} \tag{4}$$

by induction.

b) In the lecture, we have shown that the state before measurement is given by

$$|\psi\rangle = \frac{1}{2^N} \sum_{k,x=0}^{2^N-1} e^{2\pi i k(\phi - x/2^N)} |x\rangle \otimes |u\rangle \tag{5}$$

for eigenstate $|u\rangle$ for phase ϕ. Show that the probability of measuring $\tilde{\phi} = \phi - \delta/2^N$ is given by

$$p(\delta) = \left| \frac{e^{2\pi i \delta} - 1}{2^N (e^{2\pi i \delta/2^N} - 1)} \right|^2 . \tag{6}$$

c) Use $|e^{i\theta} - 1| \leq 2$ and $|e^{i\theta} - 1| \geq 2|\theta|/\pi$ for $\theta \in [-\pi, \pi]$ to show that

$$p(\delta) \leq \frac{1}{4\delta^2} . \tag{7}$$

d) The possible values for δ can be written as the series

$$\delta_i = \delta_0 + i \tag{8}$$

with $i \in \{-2^{N-1}, \dots, 2^{N-1} - 1\}$ and $|\delta_0| \leq \frac{1}{2}$. The precise value of δ_0 depends on N and ϕ.

The restriction $|\phi - \tilde{\phi}| \leq 2^{-n}$ translates to $|\delta| \leq 2^{N-n}$ such that the probability of failure to measure within the given accuracy is given by

$$p_{\text{fail}} = \sum_{i \in \{-2^{N-1}, \ldots, 2^{N-1}-1\}; |\delta_0 + i| > 2^{N-n}} p(\delta_0 + i) \tag{9}$$

$$\leq \sum_{i \in \{-2^{N-1}, \ldots, 2^{N-1}-1\}; |i| > 2^{N-n}-1} p(\delta_0 + i) \tag{10}$$

$$\leq \sum_{|i| > 2^{N-n}-1} p(\delta_0 + i) \leq 2 \int_{2^{N-n}-2}^{\infty} p(\delta) d\delta \, . \tag{11}$$

Show that to achieve $p_{\text{fail}} \leq \varepsilon$, we need at least

$$N = n + \log_2 \left(2 + \frac{1}{2\varepsilon} \right) \tag{12}$$

qubits.