Chapter 8: strong coupling expansion

In this chapter we perform a strong-coupling expansion of the path integral and the expectation value of a Wilson loop

$$W_C = \mathrm{Tr}\left[\prod_{\mu_i,x_i,m_i\in C} U_{\mu_i}^{m_i}(x_i)\right], \tag{1}$$

where $m_i \in \{\dagger, 1\}$. The path C could, e.g., describe a plaquette, a rectangle, or more complicated closed paths.

We will perform the expansion using the Wilson gauge action

$$S_W = \beta \sum_x \sum_{\mu<\nu}(1 - P_{\mu\nu}(x)) \tag{2}$$

with scalar plaquette

$$P_{\mu\nu}(x) = \frac{1}{N_c}\mathrm{Re}\,\mathrm{Tr}\,U_{\mu\nu}(x), \tag{3}$$

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x), \tag{4}$$

and

$$\beta = \frac{2N_c}{g^2}. \tag{5}$$

The expansion in a large coupling parameter g, therefore corresponds to an expansion in a small number β.

The expectation value of a Wilson loop is then given by

$$\begin{aligned}
\langle W_C\rangle &= \frac{\int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C}U_{\mu_i}(x_i)\right]\exp\left[-\beta\sum_x\sum_{\mu<\nu}(1-P_{\mu\nu}(x))\right]}{\int d[U]\exp\left[-\beta\sum_x\sum_{\mu<\nu}(1-P_{\mu\nu}(x))\right]} &(6)\\
&= \frac{\int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C}U_{\mu_i}(x_i)\right]\exp\left[\beta\sum_x\sum_{\mu<\nu}P_{\mu\nu}(x)\right]}{\int d[U]\exp\left[\beta\sum_x\sum_{\mu<\nu}P_{\mu\nu}(x)\right]} &(7)\\
&= \int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C}U_{\mu_i}(x_i)\right]\exp\left[\beta\sum_x\sum_{\mu<\nu}P_{\mu\nu}(x)\right](1+O(\beta)) &(8)\\
&= \int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C}U_{\mu_i}(x_i)\right]\exp\left[\frac{\beta}{N_c}\sum_x\sum_{\mu<\nu}\mathrm{Re}\,\mathrm{Tr}\,U_{\mu\nu}(x)\right](1+O(\beta)) &(9)\\
&= \int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C}U_{\mu_i}(x_i)\right]\exp\left[\frac{\beta}{2N_c}\sum_x\sum_{\mu<\nu}(\mathrm{Tr}\,U_{\mu\nu}(x)+\mathrm{Tr}\,U_{\mu\nu}^\dagger(x))\right](1+O(\beta)) &(10)\\
&= \int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C}U_{\mu_i}(x_i)\right]\exp\left[\frac{\beta}{2N_c}\sum_x\sum_{\mu\neq\nu}\mathrm{Tr}\,U_{\mu\nu}(x)\right](1+O(\beta)) &(11)\\
&= \int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C}U_{\mu_i}(x_i)\right]\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{\beta}{2N_c}\right)^n(\sum_P\mathrm{Tr}\,U_P)^n(1+O(\beta)), &(12)
\end{aligned}$$

where in the last line we sum over all possible plaquettes P (every spacetime-point and every x).

Let us now first consider the leading order $n=0$ term for which we need to perform the integral

$$\int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C}U_{\mu_i}(x_i)\right]. \tag{13}$$

Note that for any closed Wilson loop, each link $U_\mu(x)$ or its adjoint only appears once. In the case of the plaquette, e.g.,

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x) \tag{14}$$

such that if we integrate over $U_\mu(x)$ first, we are performing an integral of the type

$$M_{ab} = \int d[U_\mu(x)](U_\mu(x))_{ab}. \tag{15}$$

One can show that for any $V\in$ SU(3), we have

$$(VM)_{ab} = \int d[U_\mu(x)](VU_\mu(x))_{ab} = \int d[U_\mu(x)](U_\mu(x))_{ab} = M_{ab} \tag{16}$$

due to the invariance of the Haar measure. In matrix form, we have

$$VM = M \tag{17}$$

for any group element V. One can show that this can only be satisfied for

$$M = 0. \tag{18}$$

The leading term $n = 0$ therefore vanishes.

Before moving on to the $n = 1$ term, we first verify the integral equation numerically by a straightforward extension of our Haar integration routine to SU(3).

In [90]:

```
import gpt as g
import numpy as np
import scipy.linalg

su3_generators = g.object_type.ot_matrix_su_n_fundamental_algebra(3).generators(np.complex128)

U = [np.matrix(scipy.linalg.expm(2j*sum([np.random.normal()*p.array for p in su3_generators]))) for i in range(40)]
for u in U:
    assert np.linalg.norm(u * u.H - np.identity(3)) < 1e-14

np.random.seed(13)

def haar_integrate(f, U, N, NskipToMakeIndependent):
    u0 = U[np.random.randint(0,len(U))]
    val = 0.0
    val2 = 0.0
    n = 0.0
    for i in range(N):
        v = f(u0).real
        val += v
        val2 += v**2
        n += 1
        for j in range(NskipToMakeIndependent):
            u0 = U[np.random.randint(0,len(U))] * u0
    # just use biased estimator for error
    return (val/n, (val2/n - (val/n)**2)**0.5 / n**0.5 )

for a in range(3):
    for b in range(3):
        print(f"\int dU U_{a}{b}",haar_integrate(lambda u: u[a,b], U, 10000, 2))
```

```
\int dU U_00 (0.012514483442892249, 0.0040934094282299)
\int dU U_01 (0.006719000502069609, 0.004046532911347319)
\int dU U_02 (-0.001452925711356024, 0.004136127321453077)
\int dU U_10 (0.0009426477768964693, 0.0040616879424338545)
\int dU U_11 (-0.00044307235985169323, 0.004088854622644644)
\int dU U_12 (-0.002986773309435577, 0.004102720114800084)
\int dU U_20 (-0.005149322584176619, 0.004073162320270486)
\int dU U_21 (-0.00013018166498728664, 0.004102016198185097)
\int dU U_22 (-0.0015228645553870606, 0.004093229510335719)
```

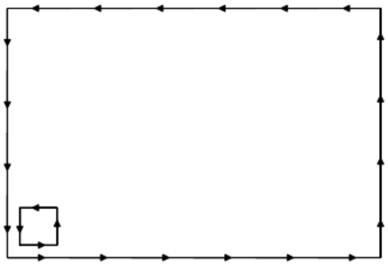
We now consider the next term in the strong coupling expansion, i.e., the $n = 1$ term. We have to compute

$$\int d[U]\mathrm{Tr}\left[\prod_{\mu_i,x_i\in C} U_{\mu_i}(x_i)\right]\frac{\beta}{2N_c}\sum_P \mathrm{Tr}\, U_P(1+O(\beta))\,. \tag{19}$$

Let us for concreteness look at the term, where P is the counter-clockwise plaquette at the origin in the $0-1$ plane and that we consider a wilson loop in the $0-1$ plane with size 6×4. We plot this scenario below:

In [53]:

```
import matplotlib.pyplot as plt

def draw_U(x,mu):
    plt.arrow(x[0],x[1],mu[0],mu[1])
    plt.arrow(x[0],x[1],mu[0]*0.5,mu[1]*0.5, head_width=0.1, head_length=0.1, fc='k', ec='k')

def draw_P(x):
    a = 0.2; oma = 1.0 - a; l = 1.0 - 2*a;
    draw_U([x[0]+a,x[1]+a],[l,0.0])
    draw_U([x[0]+oma,x[1]+oma],[-l,0.0])
    draw_U([x[0]+oma,x[1]+a],[0.0,l])
    draw_U([x[0]+a,x[1]+oma],[0.0,-l])

fig, ax = plt.subplots()

plt.axis("off")
ax.set_aspect('equal')

nx = 6
ny = 4
plt.xlim(-0.1,nx + 0.1)
plt.ylim(-0.1,ny + 0.1)

for x in range(nx):
    draw_U([x,0],[1,0])
    draw_U([x+1,ny],[-1,0])
for y in range(ny):
    draw_U([0,y+1],[0,-1])
    draw_U([nx,y],[0,1])

draw_P([0,0])

plt.show()
```



In this case there are a lot of links that only appear once and therefore their integrals vanish. In addition, however, there are some links that appear twice as in the integral

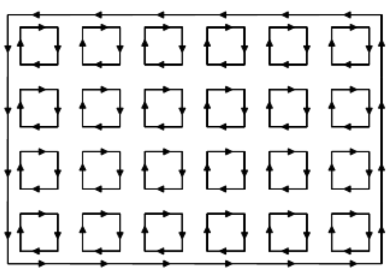
$$M^{(2)}_{ab,cd} = \int d[U_\mu(x)](U_\mu(x))_{ab}(U_\mu(x))_{cd}\,. \tag{20}$$

One can show that such integrals have to vanish. We check this again numerically.

In [94]:
```
np.random.seed(13)
for a in range(3):
    for b in range(3):
        for c in range(3):
            for d in range(3):
                v, e = haar_integrate(lambda u: u[a,b]*u[c,d], U, 4000, 2)
                if abs(v) >= 3*e:
                    print(a,b,c,d, v, e)
```

We see that the configurations with minimal power of β that can survive are those where each link in W_C is matched by the inverse link of a given plaquette. The corresponding configuration is plotted below.

In [66]:
```
fig, ax = plt.subplots()

plt.axis("off")
ax.set_aspect('equal')

def draw_invP(x):
    a = 0.2; oma = 1.0 - a; l = 1.0 - 2*a;
    draw_U([x[0]+oma,x[1]+a],[-l,0.0])
    draw_U([x[0]+a,x[1]+oma],[l,0.0])
    draw_U([x[0]+oma,x[1]+oma],[0.0,-l])
    draw_U([x[0]+a,x[1]+a],[0.0,l])

nx = 6
ny = 4
plt.xlim(-0.1,nx + 0.1)
plt.ylim(-0.1,ny + 0.1)

for x in range(nx):
    draw_U([x,0],[1,0])
    draw_U([x+1,ny],[-1,0])
for y in range(ny):
    draw_U([0,y+1],[0,-1])
    draw_U([nx,y],[0,1])

for x in range(nx):
    for y in range(ny):
        draw_invP([x,y])

plt.show()
```



In this configuration, we only have to consider integrals of the type

$$N(X)_{ab} = \int d[U_\mu(x)](U_\mu(x)XU_\mu(x)^\dagger)_{ab} \tag{21}$$

for which we can show that for every group element V, we have

$$(VN(X))_{ab} = \int d[U_\mu(x)](VU_\mu(x)XU_\mu(x)^\dagger)_{ab} = \int d[U_\mu(x)](U_\mu(x)XU_\mu(x)^\dagger V)_{ab} = (N(X)V)_{ab}\,. \tag{22}$$

We therefore have that $N(X)$ commutes with every group element V and therefore by Schur's lemma it must be proportional to the identity matrix, i.e.,

$$N(X)_{ab} = \int d[U_\mu(x)](U_\mu(x)XU_\mu(x)^\dagger)_{ab} = n(X)\delta_{ab}\,. \tag{23}$$

We also have that $N(X)$ must be linear in X such that we must be able to write

$$n(X) = n_{ab} X_{ba} \,.$$

Since this needs to hold for arbitrary X, we now have the identity

$$\int d[U_\mu(x)] (U_\mu(x))_{ab} (U_\mu(x)^\dagger)_{cd} = n_{bc} \delta_{ad} \,. \tag{24}$$

We now determine n_{bc} by setting $a = d$ and summing over them such that we have

$$\delta_{bc} = n_{bc} N_c \,. \tag{25}$$

We therefore finally arrive at the most important integration formula for this chapter

$$\int d[U_\mu(x)] (U_\mu(x))_{ab} (U_\mu(x)^\dagger)_{cd} = \frac{1}{N_c} \delta_{bc} \delta_{ad} \,. \tag{26}$$

Let us first verify this numerically:

In [95]:

```
np.random.seed(13)
for a in range(3):
    for b in range(3):
        print(f"checking all a={a}, b={b}")
        for c in range(3):
            for d in range(3):
                v, e = haar_integrate(lambda u: u[a,b]*u.H[c,d], U, 2000, 2)
                err = abs(v - (1.0/3.0 if b == c and a == d else 0.0))
                assert err < 3*e
```

```
checking all a=0, b=0
checking all a=0, b=1
checking all a=0, b=2
checking all a=1, b=0
checking all a=1, b=1
checking all a=1, b=2
checking all a=2, b=0
checking all a=2, b=1
checking all a=2, b=2
```

The lowest-order β contribution for a $n_s \times n_t$ Wilson loop is therefore given by

$$\langle W_C \rangle = \int d[U] \mathrm{Tr} \left[\prod_{\mu_i, x_i \in C} U_{\mu_i}(x_i) \right] \frac{1}{(n_s n_t)!} \left(\frac{\beta}{2N_c} \right)^{n_s n_t} (\sum_P \mathrm{Tr}\, U_P)^{n_s n_t} (1 + O(\beta)) \,, \tag{27}$$

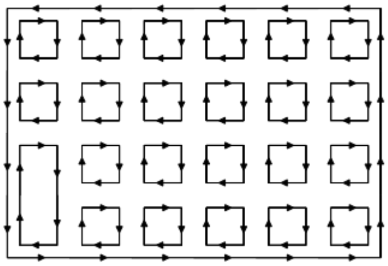
We now need to count how many link pairs we have in this case. We first note that we have $(n_s n_t)!$ permutations of the individual plaquettes. Let us now first integrate over the internal plaquette pairs. We can write such a pair as

$$\int d[U_\mu(x)] \mathrm{Tr}\, [A U_\mu(x)] \mathrm{Tr}\, [B U_\mu(x)^\dagger] = \frac{1}{N_c} \mathrm{Tr}\, [BA] \,.$$

Therefore for each removed link we get a factor of $\frac{1}{N_c}$ and obtain a joint loop with the integrated link removed. This could look like:

In [79]:

```
fig, ax = plt.subplots()

plt.axis("off")
ax.set_aspect('equal')

def draw_invP_2(x):
    a = 0.2; oma = 1.0 - a; l = 1.0 - 2*a;
    tma = 2.0 - a; tl = 1.0 - a;
    draw_U([x[0]+oma,x[1]+a],[-l,0.0])
    draw_U([x[0]+a,x[1]+tma],[l,0.0])
    draw_U([x[0]+oma,x[1]+tma-tl],[0.0,-tl])
    draw_U([x[0]+oma,x[1]+tma],[0.0,-tl])
    draw_U([x[0]+a,x[1]+a],[0.0,tl])
    draw_U([x[0]+a,x[1]+tl+a],[0.0,tl])

nx = 6
ny = 4
plt.xlim(-0.1,nx + 0.1)
plt.ylim(-0.1,ny + 0.1)

for x in range(nx):
    draw_U([x,0],[1,0])
    draw_U([x+1,ny],[-1,0])
for y in range(ny):
    draw_U([0,y+1],[0,-1])
    draw_U([nx,y],[0,1])

for x in range(nx):
    for y in range(ny):
        if x != 0 or y >= 2:
            draw_invP([x,y])
draw_invP_2([0,0])

plt.show()
```



Combining the factorial and $1/N_c$ powers, we find for the lowest-order β contribution for a $n_s \times n_t$ Wilson loop

$$\langle W_C \rangle = \left(\frac{\beta}{2N_c^2}\right)^{n_s n_t} \mathrm{Tr}\,[1](1+O(\beta)) \tag{28}$$

$$= N_c \exp\left(n_s n_t \log\left(\frac{\beta}{2N_c^2}\right)\right)(1+O(\beta))\,, \tag{29}$$

If we now again identify the Wilson loop with the static quark potential $V(n_s)$, we find

$$\langle W_C \rangle \propto \exp(-n_t V(n_s)) \tag{30}$$

for sufficiently large n_t. We can therefore at leading order in β identify

$$V(n_s) = -n_s \log\left(\frac{\beta}{2N_c^2}\right). \tag{31}$$

We indeed find a linear growth of the potential in distance n_s. The strong-beta expansion converges quite poorly such that the numerical value of the resulting string tension is not significant, however, we can read off

$$\sigma = -\log\left(\frac{\beta}{2N_c^2}\right) \tag{32}$$

which for our coarsest simulation at $\beta = 5.5$ corresponding to $a = 0.25$ fm would yield

$$\sigma = -\log\left(\frac{\beta}{2N_c^2}\right) \approx 1.2 \times \frac{0.1973 \text{ GeV fm}}{(0.25 \text{ fm})^2} \approx 3.8 \frac{\text{GeV}}{\text{fm}} \tag{33}$$

quite a bit larger than the approximate 1 GeV / fm found numerically in the continuum limit of pure QCD in the last chapter.