Universität Regensburg, Institut für Theoretische Physik Prof. Dr. Christoph Lehner, Andreas Hackl Summer 2020

Exercise: Quantum Computing Problem set 7 (to be discussed in week of June 15, 2020)

Problem 1 Stochastic energy fluctuations and decoherence times

In the lecture, we have discussed the Ramsey decoherence experiment for which the probability to measure a zero bit is given by

$$P_0^{\rm R}(t) = \frac{1}{2} + \frac{1}{2} \cos\left(\int_0^t d\tau \Delta E(\tau)\right) \tag{1}$$

and the Hahn echo decoherence experiment for which this probability is

$$P_0^{\rm H}(t) = \frac{1}{2} + \frac{1}{2} \cos\left(\int_0^{t/2} d\tau \Delta E(\tau) - \int_{t/2}^t d\tau \Delta E(\tau)\right) \,. \tag{2}$$

In both cases, $\Delta E(\tau)$ shall denote the precise energy splitting between the $|0\rangle$ and $|1\rangle$ eigenstates of the system for time τ . Fluctuations in this quantity introduce decoherence in the qubit. In this problem, we derive the quoted real probability distribution that defined the decoherence times T_2 and T_2^* from a simple model.

Consider the ansatz for the time-dependent energy splitting

$$\Delta E(\tau) \equiv \Delta \overline{E} + \eta(\tau) \tag{3}$$

with real positive $\Delta \overline{E}$ and a noise source η that we set to be constant for small time intervals Δt . Concretely, we define

$$\eta(i\Delta t + \Delta\tau) \equiv \frac{\eta_i}{\Delta t} \tag{4}$$

with integer *i* and $0 \leq \Delta \tau < \Delta t$. So for $0 \leq \tau < \Delta t$ we have $\eta(\tau) = \eta_0$, for $\Delta t \leq \tau < 2\Delta t$ we have $\eta(\tau) = \eta_1$, and so forth.

To complete the model, we demand that η_i and η_j are statistically independent for $i \neq j$ and that we have the expectation values

$$\langle \eta_i^{2n+1} \rangle_{\eta_i} = 0, \qquad \langle \eta_i^2 \rangle_{\eta_i} = \frac{2\Delta t}{T}, \qquad \langle \eta_i^{2n+2} \rangle_{\eta_i} = O(\Delta t)^2 \tag{5}$$

for all *i* with $n \in \{0, 1, ...\}$ and positive real *T*. In practice, η_i could, e.g., be drawn from a Gaussian distribution with center 0 and variance $2\Delta t/T$, however, the actual distribution is not important as long as Eq. (5) is satisfied.

Using the following steps, show that in this model $T_2 = T_2^* = T$.

a) Show that $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ for real a and b. You may use the relation of the trigonometric functions to the complex exponential function to do this.

b) Show that

$$\phi_i^{\rm R} \equiv \int_0^{i\Delta t} d\tau \Delta E(\tau) = i\Delta t \Delta \overline{E} + \sum_{j=0}^{i-1} \eta_j \tag{6}$$

and

$$\phi_i^{\rm H} \equiv \int_0^{i\Delta t} d\tau \Delta E(\tau) - \int_{i\Delta t}^{2i\Delta t} d\tau \Delta E(\tau) = \sum_{j=0}^{i-1} \eta_j - \sum_{j=i}^{2i-1} \eta_j \,. \tag{7}$$

c) Show that

$$\langle \cos(\phi + \eta_i) \rangle_{\eta_i} = \cos(\phi) \left(1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)$$
 (8)

where ϕ is a number that does not depend on η_i . Hint: consider the Taylor expansion of $\sin(a)$ for a real a.

d) Show that

$$\langle P_0^{\mathrm{R}}(t) \rangle_{\eta_0,\eta_1,\dots} = \frac{1}{2} + \frac{1}{2} \cos(t\Delta \overline{E}) \left(1 - \frac{\Delta t}{T} + O(\Delta t)^2\right)^{t/\Delta t} \tag{9}$$

and

$$\langle P_0^{\rm H}(2t) \rangle_{\eta_0,\eta_1,\dots} = \frac{1}{2} + \frac{1}{2} \left(1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)^{2t/\Delta t}$$
(10)

for all t that can be written as $t = i\Delta t$ for integer $i \ge 0$.

e) Show that

$$\lim_{\Delta t \to 0} \left(1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)^{t/\Delta t} = e^{-t/T} \,. \tag{11}$$

It may be helpful to consider the logarithm of both sides.

f) Show that by taking the limit of $\Delta t \to 0$, we find

$$\langle P_0^{\rm R}(t) \rangle_{\eta_0,\eta_1,\dots} = \frac{1}{2} + \frac{1}{2} \cos(t\Delta \overline{E}) e^{-t/T}$$
 (12)

and

$$\langle P_0^{\rm H}(t) \rangle_{\eta_0,\eta_1,\dots} = \frac{1}{2} + \frac{1}{2}e^{-t/T}$$
 (13)

for any real $t \ge 0$ and therefore $T_2 = T_2^* = T$.