Problem 1  Stochastic energy fluctuations and decoherence times

In the lecture, we have discussed the Ramsey decoherence experiment for which the probability to measure a zero bit is given by

$$ P^R_0(t) = \frac{1}{2} + \frac{1}{2} \cos \left( \int_0^t d\tau \Delta E(\tau) \right) $$

(1)

and the Hahn echo decoherence experiment for which this probability is

$$ P^H_0(t) = \frac{1}{2} + \frac{1}{2} \cos \left( \int_0^{t/2} d\tau \Delta E(\tau) - \int_{t/2}^t d\tau \Delta E(\tau) \right). $$

(2)

In both cases, $\Delta E(\tau)$ shall denote the precise energy splitting between the $|0\rangle$ and $|1\rangle$ eigenstates of the system for time $\tau$. Fluctuations in this quantity introduce decoherence in the qubit. In this problem, we derive the quoted real probability distribution that defined the decoherence times $T_2$ and $T^*_2$ from a simple model.

Consider the ansatz for the time-dependent energy splitting

$$ \Delta E(\tau) \equiv \Delta \overline{E} + \eta(\tau) $$

(3)

with real positive $\Delta \overline{E}$ and a noise source $\eta$ that we set to be constant for small time intervals $\Delta t$. Concretely, we define

$$ \eta(i\Delta t + \Delta \tau) \equiv \frac{\eta_i}{\Delta t} $$

(4)

with integer $i$ and $0 \leq \Delta \tau < \Delta t$. So for $0 \leq \tau < \Delta t$ we have $\eta(\tau) = \eta_0$, for $\Delta t \leq \tau < 2\Delta t$ we have $\eta(\tau) = \eta_1$, and so forth.

To complete the model, we demand that $\eta_i$ and $\eta_j$ are statistically independent for $i \neq j$ and that we have the expectation values

$$ \langle \eta_i^{2n+1} \rangle_{\eta_i} = 0, \quad \langle \eta_i^{2n} \rangle_{\eta_i} = \frac{2\Delta t}{T}, \quad \langle \eta_i^{2n+2} \rangle_{\eta_i} = O(\Delta t)^2 $$

(5)

for all $i$ with $n \in \{0, 1, \ldots\}$ and positive real $T$. In practice, $\eta_i$ could, e.g., be drawn from a Gaussian distribution with center 0 and variance $2\Delta t/T$, however, the actual distribution is not important as long as Eq. (5) is satisfied.

Using the following steps, show that in this model $T_2 = T^*_2 = T$.

a) Show that $\cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b)$ for real $a$ and $b$. You may use the relation of the trigonometric functions to the complex exponential function to do this.
b) Show that
\[ \phi_R^i \equiv \int_0^{i\Delta t} d\tau \Delta E(\tau) = i\Delta t \Delta E + \sum_{j=0}^{i-1} \eta_j \]  
and
\[ \phi_H^i \equiv \int_0^{i\Delta t} d\tau \Delta E(\tau) - \int_{i\Delta t}^{2i\Delta t} d\tau \Delta E(\tau) = \sum_{j=0}^{i-1} \eta_j - \sum_{j=i}^{2i-1} \eta_j. \]

c) Show that
\[ \langle \cos(\phi + \eta_i) \rangle_{\eta_i} = \cos(\phi) \left( 1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right) \]  
where \( \phi \) is a number that does not depend on \( \eta_i \). Hint: consider the Taylor expansion of \( \sin(a) \) for a real \( a \).

d) Show that
\[ \langle P^R_0(t) \rangle_{\eta_0,\eta_1,...} = \frac{1}{2} + \frac{1}{2} \cos(t\Delta E) \left( 1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)^{t/\Delta t} \]  
and
\[ \langle P^H_0(2t) \rangle_{\eta_0,\eta_1,...} = \frac{1}{2} + \frac{1}{2} \left( 1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)^{2t/\Delta t} \]  
for all \( t \) that can be written as \( t = i\Delta t \) for integer \( i \geq 0 \).

e) Show that
\[ \lim_{\Delta t \to 0} \left( 1 - \frac{\Delta t}{T} + O(\Delta t)^2 \right)^{t/\Delta t} = e^{-t/T}. \]  
It may be helpful to consider the logarithm of both sides.

f) Show that by taking the limit of \( \Delta t \to 0 \), we find
\[ \langle P^R_0(t) \rangle_{\eta_0,\eta_1,...} = \frac{1}{2} + \frac{1}{2} \cos(t\Delta E)e^{-t/T} \]  
and
\[ \langle P^H_0(2t) \rangle_{\eta_0,\eta_1,...} = \frac{1}{2} + \frac{1}{2} e^{-t/T} \]  
for any real \( t \geq 0 \) and therefore \( T_2 = T^*_2 = T \).