Problem 1 Normalizer and centralizer

In the lecture, we defined the normalizer
\[ N(S) \equiv \{ E \in G_n | E g E^\dagger \in S, \forall g \in S \} . \] (1)

Show that \( Z(S) = N(S) \) if \( S \) does not contain \(-1\) with centralizer of \( S \) in \( G_n \) defined as
\[ Z(S) \equiv \{ E \in G_n | E g = g E, \forall g \in S \} . \] (2)

Hint: first show that \( Z(S) \subseteq N(S) \) by showing every element of \( Z(S) \) is in \( N(S) \). Therefore if \( N(S) \neq Z(S) \), there is a \( E \in N(S) \) with \( E \notin Z(S) \). Show that for such an \( E \), we have \( E g E^\dagger = -g \in S \) for a \( g \in S \). Then since \( S \) is a group, \( g^{-1} \in S \) and therefore \( -gg^{-1} = -1 \in S \).

Problem 2 Stabilizer code [5,1]

Consider the [5,1] stabilizer code defined by the generators
\[ K_1 = X_1 Z_2 Z_3 X_4, \quad K_2 = X_2 Z_3 Z_4 Z_5, \] (3)
\[ K_3 = X_1 X_3 Z_4 Z_5, \quad K_4 = Z_1 X_2 X_4 Z_5 \] (4)
and
\[ Z = \prod_{i=1}^{5} Z_i . \] (5)

a) Construct the logical states \( |0_L\rangle \) and \( |1_L\rangle \) using the method described in the lecture of July 9.

b) Using the result of problem 1, show that this code corrects against \( X_l, Z_l, \) and \( X_l Z_l \) errors for \( l \in \{1, 2, 3, 4, 5\} \). It therefore corrects for arbitrary one-qubit errors. You may restrict yourself to show this only for the error set \( \{1, X_1, Z_1, X_1 Z_1\} \).

Hint: from problem 1 it suffices to show that for any two errors \( E_i, E_j \) the product \( E_i^\dagger E_j \) is either a product of generators or does not commute with at least one generator.

c) Using the construction of the lecture on July 9, create a circuit that corrects for a single such error. (Use 4 ancilla bits to store the syndromes \( \beta_l \) and apply the corresponding correction steps depending on the ancilla values.)

d) Implement this circuit in sqc (optional).