Problem 1 One-qubit depolarization channel

First show that for a density matrix $\rho$, we have

$\frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{2} = 1$ \hspace{1cm} (1)

Using this identity, derive the operator sum representation of the depolarization channel

$E(\rho) = \frac{1 - p}{2}I + p\rho$ \hspace{1cm} (2)

with $0 \leq p \leq 1$, i.e., find matrices $E_k$ for

$E(\rho) = \sum_k E_k\rho E_k^\dagger$ \hspace{1cm} (3)

Problem 2 Fidelity of one-qubit depolarization channel

Let $\sigma = E(\rho)$ for the one-qubit depolarization channel. Calculate the fidelity

$F(\rho, \sigma) \equiv \text{Tr} \left[ \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \right]$ \hspace{1cm} (4)

for a pure initial state $\rho = |\Psi \rangle \langle \Psi |$. You should find that the fidelity is independent of the state $\rho$.

Problem 3 Fidelity of phase decoherence channel

Let $\sigma = E(\rho)$ for the one-qubit phase decoherence channel defined through Kraus operators

$E_0 = \sqrt{\alpha}1$ \hspace{1cm} $E_1 = \sqrt{1 - \alpha}Z$ \hspace{1cm} (5)

Calculate the fidelity $F(\rho, \sigma)$ for a pure initial state $\rho = |\Psi \rangle \langle \Psi |$. In this case the fidelity will depend on $|\Psi \rangle$ such that it is useful to define

$F_{\text{min}} = \min_{|\Psi \rangle} F$ \hspace{1cm} (6)

as the fidelity minimized over all possible states $|\Psi \rangle$. What is $F_{\text{min}}$?