

## Exercise: Quantum Computing

### Problem set 8 (to be discussed in week of June 24, 2019)

#### Problem 1 Operator sum representation of CNOT

The first entangled state that we discussed in the lecture was the Bell state that we prepared by a Hadamard followed by a CNOT gate on a two-qubit system. We will now consider a two-qubit system with least-significant qubit (0) being the principal system and the most-significant qubit (1) being the environment and describe the entangling CNOT in the operator sum representation.

Consider a system that originally is in the unentangled state  $\rho \equiv \rho_P \otimes \rho_E$  with density matrices for the principal system  $\rho_P$  and for the environment  $\rho_E$ . Without loss of generality, you may assume  $\rho_E = |e_0\rangle\langle e_0|$ , i.e., the environment starts in a pure state. (If it were not, you could use the purification technique described in the lecture.)

Let  $\text{CNOT}(i, j)$  be the CNOT gate with control bit  $i$  and target bit  $j$  and

$$\mathcal{E}_U(\rho_P) \equiv \text{Tr}_E \left[ U \rho U^\dagger \right]. \quad (1)$$

Find the operator sum representation, i.e., find operators  $E_k$  for which

$$\mathcal{E}_U(\rho_P) = \sum_k E_k \rho_P E_k^\dagger \quad (2)$$

for  $U = \text{CNOT}(0,1)$  as well as  $U = \text{CNOT}(1,0)$ . Give explicit matrix representations of the  $E_k$ . It may be helpful to first write out the matrix representation of  $U = U_{ab;cd} |ab\rangle\langle cd|$  with sum over repeated indices implied.

#### Problem 2 Generalized amplitude damping

Consider the generalized amplitude damping channel described by Kraus operators  $E_k$  in

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad (3)$$

with

$$E_0 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \sqrt{p} \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad (4)$$

$$E_2 = \sqrt{1-p} X \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} X, \quad E_3 = \sqrt{1-p} X \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} X, \quad (5)$$

NOT matrix  $X$ , and  $\gamma \equiv 1 - e^{-t/T_1}$ , and  $p \in [0, 1]$ . This channel can describe the energy relaxation process with relaxation time  $T_1$  at finite temperature  $T$ :

a) We can define a discrete time-series of density matrices  $\rho_n$  defined by

$$\rho_{n+1} = \mathcal{E}(\rho_n) \quad (6)$$

given a starting matrix  $\rho_0$ . One can show that at sufficiently large  $n$  for a general  $\rho_0$ , the system stabilizes to

$$\rho_\infty = \mathcal{E}(\rho_\infty). \quad (7)$$

Find  $\rho_\infty$ .

- b) At finite temperature we expect a Boltzmann distribution  $p_n = e^{-E_n/(k_B T)}/\mathcal{Z}$  with  $\mathcal{Z} = \sum_n e^{-E_n/(k_B T)}$  with energies  $E_n$  of states  $n$  and Boltzmann constant  $k_B$ . In our case we have two states  $|0\rangle$  and  $|1\rangle$  with energies  $E_0$  and  $E_1$ . What temperature does  $\rho_\infty$  correspond to?