Universität Regensburg, Institut für Theoretische Physik
Prof. Dr. Christoph Lehner, Simon Bürger Summer 2019

Exercise: Quantum Computing
Problem set 5 (to be discussed in week of June 3, 2019)

Problem 1 Phase estimation with a modest number of qubits

Consider the phase estimation algorithm using N qubits to discretize the approximation $\tilde{\phi}$ of the true value ϕ for an eigenvalue

$$e^{2\pi i\phi} \tag{1}$$

of a unitary matrix U. In this case, the possible measured values are

$$\tilde{\phi} = \frac{x}{2^N} \tag{2}$$

with $x \in \{0, 1, \ldots, 2^N - 1\}$. In problem set 4, we have established a lower bound on the probability to measure $\tilde{\phi}$ within a certain interval of ϕ. Using this result, we know that in the large N limit we can resolve ϕ to arbitrary precision with high probability, i.e., for sufficiently large N, we will only need few measurements to obtain a precise bound on ϕ.
In practice, the number of available qubits is modest but we may be able to perform many measurements. We will now explore how to improve the phase estimation algorithm in this limit.

a) Using the result

$$p(\delta) = \left| \frac{e^{2\pi i\delta} - 1}{2^N (e^{2\pi i\delta/2^N} - 1)} \right|^2 \tag{3}$$

with $\tilde{\phi} = \phi - \delta/2^N$ derived in problem set 4, show that

$$p(\delta) = \frac{\sin(\delta\pi)^2}{(\delta\pi)^2} \left(1 + \frac{1}{3}(\delta\pi)^2\varepsilon^2 + O(\varepsilon^4) \right) \tag{4}$$

with $\varepsilon = 2^{-N}$. Show that even for the worst case of $|\delta| = \frac{1}{2}$, the correction of the ε^2 term is smaller than one percent of the $\varepsilon = 0$ value already for $N = 4$.

b) Consider that most measurements will return a value of $\tilde{\phi} = \tilde{\phi}_0$. Show that the ratio r of number of measurements of $\tilde{\phi}_0 - \varepsilon$ divided by the number of measurements of $\tilde{\phi}_0 + \varepsilon$ in the limit of infinitely many measurements is given by

$$r = \frac{(\delta - 1)^2}{(\delta + 1)^2} + O(\varepsilon^2)\,. \tag{5}$$

c) Show that by measuring r precisely, we can obtain a correction

$$\phi = \tilde{\phi}_0 + \varepsilon \frac{1 - \sqrt{r}}{1 + \sqrt{r}} + O(\varepsilon^3)\,. \tag{6}$$

d) In the lecture we have used the quantum simulator to study a matrix with two eigenvalues for $\phi = \frac{1}{4}, \frac{3}{4}$ for $N = 4$. You may assume that you have measured 1000000 times and obtained

$$N(0.3125) = 23977\,, \qquad N(0.375) = 286983\,, \qquad N(0.4375) = 127876 \tag{7}$$

with $N(\tilde{\phi})$ giving the number of measurements of the specific value of $\tilde{\phi}$. Use the derived correction to obtain an improved estimate for the close eigenvalue ϕ.

e) Derive the correction of order ε^3 to ϕ using only r and use it to further improve the numerical estimate of d). To this end it is useful to explicitly write

$$\delta = \delta_0 + \varepsilon^2 \delta_1 \tag{8}$$

and then solve for δ_0 and δ_1 in a version of Eq. (5) which includes $O(\varepsilon^2)$ corrections. You should find

$$\frac{\delta_1}{\delta_0} = \frac{1}{3}\pi^2(1 - \delta_0^2)\,. \tag{9}$$

and therefore

$$\phi = \tilde{\phi}_0 + \varepsilon\delta_0 \left(1 + \frac{1}{3}\varepsilon^2\pi^2(1 - \delta_0^2)\right) + O(\varepsilon^5) \tag{10}$$

with

$$\delta_0 = \frac{1 - \sqrt{r}}{1 + \sqrt{r}}\,. \tag{11}$$

Problem 2 Numerical example (optional)

Based on the phase.ipynb notebook in the simulator, perform the improved estimation of problem 1 for $N = 4$ and a unitary matrix with two eigenvalues for $\phi = 0.3, 0.7$.