Problem 1  Matrix representation of CNOT

In the lecture we showed that the matrix representation of

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]  \hspace{1cm} (1)

is

Remember that we draw the least-significant qubit at the top and use the standard basis \( |00\rangle, |01\rangle, |10\rangle, |11\rangle \). Derive the matrix representation for

in the same basis.

Problem 2  Inverse of CNOT

Show that

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}^{-1}
\]  \hspace{1cm} (2)

Problem 3  Relation of SWAP and CNOT

Show that

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]  \hspace{1cm} (3)

Problem 4  Universal one qubit gates

Show that the \( H \) and \( R_\phi \) gates can generate an arbitrary two-dimensional unitary matrix. You may first show

\[
X = H R_\pi H
\]  \hspace{1cm} (4)
and then restrict yourself to infinitesimal unitary matrices of the form

\[ U = 1 + i\varepsilon M + O(\varepsilon^2) \]  

with Hermitian matrix \( M^\dagger = M \) and \( \varepsilon \in \mathbb{R} \). Show that

\[ U_1 = HR_{2x}H, \quad U_2 = R_{\pi/2}HR_{2y}HR_{-\pi/2}, \quad U_3 = R_{\varepsilon z}, \quad U_4 = X R_{\varepsilon t}X \]  

with \( x, y, z, t \in \mathbb{R} \) are all of this form and their product maps out a general \( M \).

**Problem 5  Universal gates**

Show that adding a CNOT gate to a universal one-qubit gate is sufficient to generate an arbitrary unitary matrix in \( S^N \). Use the gate \( s^{(ij)} \) defined in the lecture on April 30 to extend the proof of Problem 4 to the general case of \( M \) in \( 2^N \) dimensions.