Universität Regensburg

Adic Pro-étale Sheaves

Master's thesis

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Abstract

This thesis aims to study I-adic pro-étale sheaves, where I is an ideal of a ring R. We give a definition and show that it is equivalent to the notion of I-adic sheaves on the étale site if I is finitely generated. Moreover, we extend the existing theory by discovering basic properties of adic pro-étale sheaves. Our main results include that for a noetherian scheme X and a noetherian ring R, I-adic pro-étale sheaves form a weak Serre subcategory of the sheaves of R-modules on X_{proet}. To avoid a technical construction of a pro-étale site, we axiomatically describe a list of properties that a pro-étale enlargement should have. The resulting theory about I-adic sheaves is then applicable to any setting fulfilling these properties.

Contents

Introduction

An important aspect in the study of étale cohomology is the examination of I-adic sheaves. They were introduced by A. Grothendiecke in the 1960s in order to obtain a cohomology theory which works over any base field. This evolved theory was finally used by A. Grothendiecke and P. Deligne to prove the Weil Conjectures. It is needless to say that I-adic sheaves are an important concept in algebraic geometry and number theory and that it is worth considering these types of objects. The definition for an I-adic sheaf, which originally appears in [8, Exposé VI], is the following.

Definition. Assume R is a ring, $I \subset R$ an ideal and X a scheme. An inverse system $(F_n)_{n\in\mathbb{N}}$ of sheaves of R-modules on the étale site X_{et} is called *I-adic system* (or by abuse of notation *I-adic sheaf*) if each F_n is constructible,

- $I^{n+1}F_n = 0$, and
- the morphism $F_{n+1} \to F_n$ gives rise to an isomorphism $F_{n+1}/I^{n+1}F_{n+1} \cong F_n$.

The *étale cohomology groups* of an I-adic system are then defined to be

$$H^i(X_{\text{et}}, (F_n)_{n\in\mathbb{N}}) := \lim_n H^i(X_{\text{et}}, F_n).$$

Unfortunately, the étale cohomology groups can in general not be computed as $H^i(X_{\text{et}}, \lim_n F_n)$. That was one of the reasons for B. Bhatt and P. Scholze to introduce a new site X_{proet}, the so called pro-étale site (see [2]). Their site comes with many advantageous properties, including an improved behavior of inverse limits. The question of how the above definition should be realized in the pro-étale world finally leads to the notion of adic pro-étale sheaves.

The idea of adic pro-étale sheaves comes from *The Stacks Project* [15, Tag 09BS]. Here, one can find the following definition.

Definition (adic pro-étale sheaves)**.** A sheaf of R-modules $\mathcal{F}$ on X_{proet} is called *constructible R-sheaf* or *I-adic (pro-étale) sheaf* if $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of R/I^nR-modules for every $n \in \mathbb{N}$ and

$$\mathcal{F} \cong \lim_n \mathcal{F}/I^n\mathcal{F}.$$

Although this seems to be a promising definition, the literature does not encompass basic properties of adic pro-étale sheaves. This thesis aims to fill this gap. We want to provide fundamental statements concerning these objects and thereby extending the existing theory. Below, we present some results from this thesis, which are not yet covered in the literature. Assume I is finitely generated.

- The category of I-adic systems on X_{et} is in equivalence with the category of I-adic sheaves on X_{proet}.
- Adic pro-étale sheaves are stable under extension by zero along a quasi-compact open subscheme.

If moreover X and R are noetherian, then

- The category of I-adic pro-étale sheaves is an abelian subcategory of the category $\mathrm{Mod}_R(X_{\mathrm{proet}})$ of sheaves of R-modules on X_{proet}.
  And even better:
- The category of I-adic pro-étale sheaves forms a weak Serre subcategory of the category $\mathrm{Mod}_R(X_{\mathrm{proet}})$.
- The category of I-adic sheaves on X_{proet} is noetherian.

Note that similar properties are true for I-adic étale sheaves, which highlights the relevance of these essential statements. Consequently, this thesis can be seen as justification for the definition of adic pro-étale sheaves and moreover as a foundation for further extensions of this theory.

To avoid a technical construction of a pro-étale site, we axiomatically describe a list of properties that a pro-étale enlargement should have. The advantage of an axiomatic viewpoint is that it is applicable to any setting fulfilling the desired properties. Moreover, it can be seen as guideline for the construction of a pro-étale site.

Structure. In Chapter 1 we recall basic facts, the reader should be familiar with. Mainly, the theory of sites and sheaves is recalled and an overview over important concepts, like morphisms of sites, sheafification or restrictions, is given. Moreover, Chapter 1 also covers some statements about Serre subcategories and complete modules. Chapter 2 shortly introduces the étale site and gives some relevant properties. This first two chapters are surely standard and the ideas can be found in almost every textbook on étale cohomology. We mainly used [15] and [16] to gather all the relevant information.

The literature usually covers the theory of I-adic étale sheaves for the special case $R = \mathbb{Z}_\ell$ and $I = (\ell)$ where ℓ is a prime number which is invertible in the considered schemes. As we want to use more arbitrary rings, we generalize many statements from [5] and [6] to the case where R is noetherian, and I is any ideal of R. With a few exceptions, everything can be adapted in a straightforward manner to the more general setting. The technical proofs are discussed in Chapter 3.

Finally, Chapter 4 treats the main contents of this thesis. We formulate and explain the axioms and derive the statements about I-adic sheaves in Section 4.3 and 4.4. In the latter many properties from Chapter 3 are transported to the pro-étale case. Moreover, we compare our definition of I-adic sheaves with the derived category of constructible complexes $D_{\mathrm{cons}}(X_{\mathrm{proet}}, R)$ introduced by Bhatt and Scholze in their paper [2]. At the end, we sketch the two concrete versions of the pro-étale topology from [2] and [12].

Note that many parts of Chapter 4 are not covered in the existing literature and are results from our own research.

Conventions. We assume all rings to be commutative with one. To ensure readability, even for non-experts, we try to give many details and construct a framework for our theory. Nevertheless, it is of advantage to have some basic background knowledge in sheaf theory. At some points we use the notion of derived categories, which will be not introduced in this thesis. An excellent introduction in the theory of derived categories can be found in [17]. However, most of the statements can be understood without knowing this evolved technique.

1 Background

1.1 Sites and Sheaves

In this section we will give a small introduction in the theory of sites. We will prove statements which are particularly relevant to this thesis. For additional information we will give references. To get an overview over this field one can refer to the book *Introduction to étale Cohomology* of G. Tamme [16] or *The Stacks Project* [15, Tag 00UZ]. Most of the contents of this section come from these two sources.
Let us first give some basic definitions.

1.1.1 Definitions and Basic Properties

We basically follow [16, Chapter I].

Definition 1.1.1. Let $\mathcal{C}$ be a category. A *topology on* $\mathcal{C}$ consists of the following data. For every object $U \in \mathcal{C}$ a set $\mathrm{Cov}(U)$, the set of coverings, whose elements are families $(U_i \to U)_{i\in I}$ which fulfill the following three criteria.

T1 (base change) If $(U_i \to U)_{i\in I}$ is a covering and $V \to U$ a morphism in $\mathcal{C}$ then the base change $U_i \times_U V$ exists for all i and $(U_i \times_U V \to V)_{i\in I} \in \mathrm{Cov}(V)$.

T2 (composition) Let $(U_i \to U)_{i\in I} \in \mathrm{Cov}(U)$ and for every i let $(V_{i,j} \to U_i)_{j\in J_i} \in \mathrm{Cov}(U_i)$ then the family $(V_{i,j} \to U_i \to U)_{i,j}$ is an element of $\mathrm{Cov}(U)$

T3 (identity) The family $(\mathrm{id} : U \to U)$ is a covering, i.e. an element of $\mathrm{Cov}(U)$.

A site is a pair $(\mathcal{C}, \mathrm{Cov})$ where $\mathcal{C}$ is a category in which finite limits exist and a topology Cov on $\mathcal{C}$. If it is clear which topology is used, we often write $\mathcal{C}$ for $(\mathcal{C}, \mathrm{Cov})$.

Remark 1.1.2. Our definition of a site $(\mathcal{C}, \mathrm{Cov})$ requires the existence of finite limits in $\mathcal{C}$, which is not always assumed in the literature. However, we chose to include finite limits in our definition because they offer some advantages. For instance, finite limits ensure the exactness of the pullback along morphisms of sites and guarantee the existence of a terminal object in $\mathcal{C}$, since the terminal object is the limit of the empty diagram $\Delta : \emptyset \to \mathcal{C}$.

Definition 1.1.3. Let $(\mathcal{C}, \mathrm{Cov})$ be a site and let $\mathcal{A}$ be the category Set, Ab or Mod_R for a ring R. In general one demands that $\mathcal{A}$ is a complete and cocomplete category[1].

[1] I.e. all small limits and colimits exist.

A *presheaf* on $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{\text{opp}} \to \mathcal{A}.$$

A morphism of presheaves is a natural transformation and the resulting category of presheaves is also denoted as $\mathrm{PSh}((\mathcal{C}, \mathrm{Cov}), \mathcal{A})$.
A presheaf is called a *sheaf* if for every $U \in \mathcal{C}$ and every covering $(U_i \to U)_{i\in I} \in \mathrm{Cov}(U)$ the sequence

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

is exact. We denote $\mathrm{Sh}((\mathcal{C}, \mathrm{Cov}), \mathcal{A})$ as the full subcategory of sheaves. If it is clear wich category is used for $\mathcal{A}$, we also write $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ resp. $\mathrm{PSh}(\mathcal{C}, \mathrm{Cov})$ for the respective categories. If $\mathcal{A} = \mathrm{Ab}$ then we also denote $\mathrm{Sh}((\mathcal{C}, \mathrm{Cov}), \mathcal{A}) := \mathrm{Ab}(\mathcal{C}, \mathrm{Cov})$ and call sheaves $\mathcal{F} \in \mathrm{Ab}(\mathcal{C}, \mathrm{Cov})$ *abelian sheaves.*

At this point, we introduce some examples that will be more or less important for us. Later, we will see that the choice of a site has many effects on the corresponding category of sheaves. However, sheaves on the classical Zariski site lack some homological properties, which is why we will introduce the étale and the pro-étale site.

Example 1.1.4.

- Let X be a topological space and consider the category $\mathrm{Ouv}(X)$ of the open subsets of X. That is, objects are the open subsets in X and morphism are inclusions. A family $(U_i \hookrightarrow U)_{i\in I}$ is a covering if $\bigcup_{i\in I} U_i = U$. This construction gives the Zariski site, which is denoted by X_{Zar}.

- Let X be a scheme and consider the category of étale morphisms over X, i.e. the objects are étale morphisms $U \to X$. Define coverings to be families $(U_i \xrightarrow{f_i} U)_{i\in I}$ where all f_i are étale and $U = \bigcup_i f_i(U_i)$. This defines the étale site X_{et}, which will be investigated in Chapter 2.

- Let $\mathcal{C}$ be any category. The finest topology such that the assertion

  $$U \mapsto \mathrm{Hom}_{\mathcal{C}}(-, U)$$

  forms a sheaf is called *canonical topology on $\mathcal{C}$.* A justification and a concrete description can be found in [15, Tag 00Z9]. Any topology which is coarser then the canonical topology is called *subcanonical.* Note that in any subcanonical topology, the presheaves $\mathrm{Hom}_{\mathcal{C}}(-, U)$ are already sheaves.

- The *chaotic topology* on an arbitrary category $\mathcal{C}$ consists of coverings of the form $(V \xrightarrow{\cong} U)$. Although this construction turns every category into a site, this is of no practical use as the sheaves on the chaotic site are in equivalence with the presheaves on $\mathcal{C}$. Thus, considering sheaves do not provide any additional information.

Lemma 1.1.5. If $\mathcal{A}$ is a complete and cocomplete category[2] then $\mathrm{PSh}((\mathcal{C},\mathrm{Cov}),\mathcal{A})$ is complete and cocomplete. Further the limits and colimits in $\mathrm{PSh}(\mathcal{C},\mathrm{Cov})$ can be computed using the formulas

$$(\lim_i F_i)(U) = \lim_i F_i(U) \quad \text{and} \quad (\mathrm{colim}_i F_i)(U) = \mathrm{colim}_i F_i(U)$$

Proof. [15, Tag 00VB] □

Theorem 1.1.6 (sheafification)**.** The inclusion functor $\iota : \mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ admits a left adjoint $\mathrm{a} : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$. Additionally $\mathrm{a} \circ \iota \cong \mathrm{id}$, ι commutes with all limits, the functor a commutes with all colimits and is exact[3].

Proof. [16, Theorem 3.1.1, 3.2.1] □

Remark 1.1.7. Let $F : I \to \mathrm{Sh}(\mathcal{C})$ be a diagram, such that the limit $\lim_i F_i$ exists in $\mathrm{Sh}(\mathcal{C})$. Then Theorem 1.1.6 implies that $\iota(\lim_i F_i) = \lim_i \iota F_i$. In particular for every $U \in \mathcal{C}$ we have $\lim_i F_i(U) = (\lim_i F_i)(U)$. That is, taking sections commutes with limits. A similar argument can be applied to colimits as follows. The colimit $\mathrm{colim}_j F_j$ of a diagram $F : J \to \mathrm{Sh}(\mathcal{C})$ is the sheafification of the presheaf $U \longmapsto \mathrm{colim}_j F_j(U)$, because $\mathrm{a}(\mathrm{colim}_j F_j) = \mathrm{colim}_j \mathrm{a} F_j$ and the sheafification of a sheaf is canonically isomorphic to itself.

Corollary 1.1.8. If $\mathcal{A}$ has small limits and colimits then $\mathrm{Sh}(\mathcal{C},\mathcal{A})$ has all small limits and colimits. If further $\mathcal{A}$ is an abelian category with enough injectives, then $\mathrm{Sh}(\mathcal{C},\mathcal{A})$ is an abelian category with enough injectives. For $U \in \mathcal{C}$ there is a left exact functor

$$\begin{aligned} \Gamma(U,-) : \mathrm{Sh}(\mathcal{C}) &\longrightarrow \mathcal{A} \\ F &\longmapsto F(U). \end{aligned}$$

Proof. The claim about the injectives is [15, Tag 01DL] for $\mathcal{A} = \mathrm{Ab}$. The remaining claims directly follow from Remark 1.1.7. □

Direct and Inverse Image

In this subsections all sheaves are considered to be abelian sheaves or sheaves of sets.

Definition 1.1.9 (morphism of sites)**.** Let $(\mathcal{C},\mathrm{Cov})$ and $(\mathcal{C}',\mathrm{Cov}')$ be two sites. A *morphism of sites*

$$f : (\mathcal{C},\mathrm{Cov}) \longrightarrow (\mathcal{C}',\mathrm{Cov}')$$

consists of a functor $f^{-1} : \mathcal{C}' \to \mathcal{C}$ such that

[2] I.e. small limits and colimits exist in $\mathcal{A}$.

[3] In particular it commutes with finite limits.

1. For all elements $U' \in \mathcal{C}'$ and every covering $(U'_i \to U')_{i\in I}$ we have
$$(f^{-1}(U'_i) \to f^{-1}(U'))_{i\in I} \in \mathrm{Cov}(f^{-1}(U')).$$
2. The functor f^{-1} is left exact. That means that for any $U' \in \mathcal{C}'$, any covering $(U'_i \to U') \in \mathrm{Cov}(U')$ and any morphism $V \to U'$ the canonical morphisms
$$f^{-1}(U'_i \times_U V) \to f^{-1}(U'_i) \times_{f^{-1}(U')} f^{-1}(V)$$
are isomorphisms and f^{-1} sends the terminal object of $\mathcal{C}'$ to a terminal object of $\mathcal{C}$.

The composition of two morphisms of sites $f : \mathcal{C} \to \mathcal{C}'$ and $g : \mathcal{C}' \to \mathcal{C}''$ is carried out by the composition $(g \circ f)^{-1} := f^{-1} \circ g^{-1}$.

Example 1.1.10. Let X and Y be two topological spaces and $f : X \to Y$ a continuous map. Then we get an induced morphism of sites $f : X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$ which is defined by
$$\begin{aligned} \mathrm{Ouv}(Y) &\longrightarrow \mathrm{Ouv}(X) \\ U &\longmapsto f^{-1}(U). \end{aligned}$$

It is a simple exercise to justify the properties.

Definition 1.1.11. Let $f : (\mathcal{C}, \mathrm{Cov}) \longrightarrow (\mathcal{C}', \mathrm{Cov}')$ be a morphism of sites. The *direct image* is defined to be
$$\begin{aligned} f_\bullet : \mathrm{PSh}(\mathcal{C}) &\longrightarrow \mathrm{PSh}(\mathcal{C}') \\ F &\longmapsto \Big(U' \mapsto F(f^{-1}(U'))\Big). \end{aligned}$$

Proposition 1.1.12. The functor $f_\bullet$ admits a left adjoint $f^\bullet : \mathrm{PSh}(\mathcal{C}') \to \mathrm{PSh}(\mathcal{C})$. This functor $f^\bullet$ is called the *inverse image* of f. Both, $f_\bullet$ and $f^\bullet$ are exact.

Proof. We provide a construction, deferring a detailed proof to the literature. Let $U \in \mathcal{C}$ be an object. We are considering a category $\mathcal{I}_U$, where the objects are pairs (U', ϕ) consisting of an object $U' \in \mathcal{C}$ and a morphism $\phi : U \to f^{-1}(U')$ in $\mathcal{C}$. A morphism of pairs $(U', \phi) \to (\tilde{U}, \psi)$ is by definition a morphism $\tau : U' \to \tilde{U}$ such that
$$\begin{tikzcd} f^{-1}(U') \arrow[rr, "f^{-1}(\tau)"] & & f^{-1}(\tilde{U}) \\ & U \arrow[ul] \arrow[ur] & \end{tikzcd}$$
commutes. Given a sheaf $\mathcal{F}$ on $\mathcal{C}'$ and a $U \in \mathcal{C}$ one can define
$$f^\bullet \mathcal{F}(U) := \mathrm{colim}_{((U',\phi) \in \mathcal{I}_U^{\mathrm{opp}})} \mathcal{F}(U').$$
This is already the correct presheaf as it is shown in [16, p.41 ff.]. $\square$

Definition 1.1.13 (direct image and inverse image)**.** We define the *inverse image* respectively the *direct image* of sheaves as

$$\begin{aligned} f^* &:= \mathrm{a} \circ f^{\bullet} \circ \iota : \mathrm{Sh}(\mathcal{C}') \to \mathrm{Sh}(\mathcal{C}), \\ f_* &:= \mathrm{a} \circ f_{\bullet} \circ \iota : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}') \quad \text{respectively.} \end{aligned}$$

Here, ι denotes the forgetful functor from the category of sheaves into the category of presheaves.

Proposition 1.1.14. In the notion of Definition 1.1.13, we have the following properties.

1. $f_{\bullet}$ and $f^{\bullet}$ are exact.
2. The functor f^* is left adjoint to f_*.
3. f_* is left exact and f^* is exact.

Proof. The first property is covered by Proposition 1.1.12 and the second is an immediate consequence. For the third part, the only nontrivial thing is to show that f^* exact. The right exactness is trivial as f^* is a left adjoint. For the left exactness note that f^* is the composition of left exact functors. □

Lemma 1.1.15. Let $f : \mathcal{C} \to \mathcal{C}'$ and $g : \mathcal{C}'' \to \mathcal{C}$ be two morphisms of sites. Then we have

$$(f \circ g)_* = f_* \circ g_* \quad \text{and} \quad (f \circ g)^* = g^* \circ f^*.$$

The corresponding statement holds for presheaves.

Proof. It is clear from the definition that $f_{\bullet} \circ g_{\bullet} = (f \circ g)_{\bullet}$. This directly implies the first claim. The second follows from the adjointness. □

1.1.2 Sheaves of Modules

In this subsection we will briefly introduce the notion of sheaves of modules. We orient to [15, Tag 03A4] and include only those definitions, which are of interest for us. Let $\mathcal{C}$ be a site and $\mathcal{O}$ a presheaf of rings on $\mathcal{C}$, i.e. an abelian presheaf such that for every $U \in \mathcal{C}$ the group $\mathcal{O}(U)$ admits a ring structure which is compatible with restrictions.

Definition 1.1.16. A *presheaf of $\mathcal{O}$-modules* is an abelian presheaf $\mathcal{F}$ on $\mathcal{C}$ together with a map of presheaves

$$m : \mathcal{O} \times \mathcal{F} \to \mathcal{F},$$

which defines a $\mathcal{O}(U)$-module structure on $\mathcal{F}(U)$ for any $U \in \mathcal{C}$. Morphisms of presheaves of $\mathcal{O}$-modules are morphisms of presheaves which are compatible with the above multiplication map, that means $f : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}$-presheaves if the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r, "(\mathrm{id}, f)"] \arrow[d, "m"] & \mathcal{O} \times \mathcal{G} \arrow[d, "m"] \\ \mathcal{F} \arrow[r, "f"] & \mathcal{G} \end{tikzcd}$$

commutes.
Assume $\mathcal{O}$ is a sheaf of rings. An abelian sheaf $\mathcal{F}$ is called *sheaf of $\mathcal{O}$-modules* if $\mathcal{F}$ interpreted as presheaf is a presheaf of $\mathcal{O}$-modules. Although it is not necessary to demand that $\mathcal{O}$ is a sheaf in this definition, we will not see or use an example where this is not the case. We denote the category of sheaves of $\mathcal{O}$-modules on $\mathcal{C}$ by $\mathrm{Mod}_{\mathcal{O}}(\mathcal{C})$.

Definition 1.1.17. A sheaf $F \in \mathrm{Sh}(\mathcal{C}, \mathcal{A})$ is called *constant with values in $E \in \mathcal{A}$* if it is the sheafification of a presheaf of the form

$$U \longmapsto E.$$

We also write $\underline{E}_{\mathcal{C}}$ for the constant sheaf on $\mathcal{C}$ with values in E. If it is clear which site we consider, one simply writes $\underline{E}$.

Lemma 1.1.18. The functor

$$\underline{\bullet} : \mathrm{Ab} \to \mathrm{Sh}(\mathcal{C}, \mathrm{Ab})$$

is exact. Further $\iota \circ \underline{\bullet}$ is exact. That means $\underline{\bullet}$ transforms exact sequences of abelian groups to exact sequences of presheaves.

Proof. The exactness of $\underline{\bullet}$ is clear since the sheafification is exact. Since the inclusion functor ι is left exact, it suffices to show that a surjection $B \to C$ gives an epimorphism $\iota\underline{B} \to \iota\underline{C}$ in the category of presheaves. Pick a set theoretical section $s : C \to B$ of $B \to C$ to get a section of presheaves of sets $\underline{s} : \iota\underline{C} \to \iota\underline{B}$. $\square$

Lemma 1.1.19. Let $\mathcal{C}$ and $\mathcal{D}$ be sites, M an abelian group or an Λ-module for a ring Λ and $\epsilon : \mathcal{C} \to \mathcal{D}$ a morphism of sites. We obtain the formula

$$\underline{M}_{\mathcal{C}} \cong \epsilon^*(\underline{M}_{\mathcal{D}})$$

Proof. We write M for the presheaf with constant values in M. As preliminary step we notice that the construction in the proof of Proposition 1.1.12 implies

$$\epsilon^{\bullet} M(U) = \lim_{U' \in I_U^{\mathrm{opp}}} M(U') = M$$

and hence $\epsilon^\bullet M = M$. For any $\mathcal{H} \in \mathrm{Sh}(\mathcal{C})$ we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\epsilon^* \underline{M}_{\mathcal{D}}, \mathcal{H}) &\cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{D})}(\underline{M}_{\mathcal{D}}, \epsilon_* \mathcal{H}) \\
&\cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(M, \epsilon_* \mathcal{H}) \\
&= \mathrm{Hom}_{\mathrm{PSh}(\mathcal{D})}(M, \epsilon_\bullet \mathcal{H}) \\
&\cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\epsilon^\bullet M, \mathcal{H}) \\
&\cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(M, \mathcal{H}) \\
&\cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\underline{M}_{\mathcal{C}}, \mathcal{H}).
\end{aligned}$$

We used that $\underline{M}_{\mathcal{C}} = \mathrm{a}(M)$ and the adjunctions

$$a : \mathrm{PSh}(\mathcal{C}) \rightleftarrows \mathrm{Sh}(\mathcal{C}) : \iota,\ \epsilon^* : \mathrm{Sh}(\mathcal{D}) \rightleftarrows \mathrm{Sh}(\mathcal{C}) : \epsilon_* \text{ and } \epsilon^\bullet : \mathrm{PSh}(\mathcal{D}) \rightleftarrows \mathrm{PSh}(\mathcal{C}) : \epsilon_\bullet.$$

This proves the claim. □

Example 1.1.20. If Λ is a ring then the sheaf $\underline{\Lambda}$ is a sheaf of rings.

Remark 1.1.21. Let $\epsilon : \mathcal{C} \to \mathcal{D}$ be a morphism of sites and let F be a sheaf of $\underline{\Lambda}_{\mathcal{D}}$-modules. Then $\epsilon^* F$ is canonically an $\epsilon^* \underline{\Lambda}_{\mathcal{D}} = \underline{\Lambda}_{\mathcal{C}}$-module. In particular, the pullback ϵ^* is a well-defined functor

$$\mathrm{Mod}_{\Lambda}(\mathcal{D}) \to \mathrm{Mod}_{\Lambda}(\mathcal{C}).$$

Moreover, it is an exercise to show that ϵ^* fulfills all the adjointness properties from the foregoing sections.

For the following let Λ be a ring.

Definition 1.1.22. A *sheaf of* Λ*-modules* $\mathcal{F}$ is defined to be a sheaf of $\underline{\Lambda}$-modules, where $\underline{\Lambda}$ is the constant sheaf associated to Λ.

Definition 1.1.23. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ is of *finite type* if there is an $n \in \mathbb{N}$ together with an epimorphism

$$\mathcal{O}^n \twoheadrightarrow \mathcal{F}.$$

Remark 1.1.24. The justification for this definition is the following. Any $\underline{\Lambda}$-module F has a Λ-module structure on $\mathcal{F}(U)$ for any $U \in \mathcal{C}$. This is defined by the canonical map

$$\underline{\Lambda}_{\mathrm{PSh}} \to \underline{\Lambda},$$

where $\underline{\Lambda}_{\mathrm{PSh}}$ denotes the constant presheaf. Conversely, if F has a sectionwise Λ-module structure which is compatible with restriction, then we get a $\underline{\Lambda}$-module structure on F by applying sheafification to the multiplication $\underline{\Lambda}_{\mathrm{PSh}} \times F \to F$. Nevertheless, one must be careful with some concepts related to these objects. For example it is not true that a finite type sheaf of $\underline{\Lambda}$-modules is of finite type as presheaf of Λ-modules. Concretely, let $X = \bigsqcup_{\mathbb{N}} \{*\}$ an infinite union of one point topological spaces and consider the Zariski site induced by X. Then the global sections of the constant sheaf are $\underline{\Lambda}(X) = \prod_{\mathbb{N}} \Lambda$, which is definitely no finite type Λ-module. Despite this, the other implication is true as sheafification commutes with finite direct sums and preserves epimorphisms.

1.1.3 Restrictions and Locally Constant Sheaves

Let $\mathcal{C}$ be a site and $U \in \mathcal{C}$ any object. We can define a category $\mathcal{C}/U$, where the objects are morphisms $V \to U$ in $\mathcal{C}$ and the morphisms are commutative triangles

$$\begin{tikzcd} V \arrow[rr] \arrow[dr] & & V' \arrow[dl] \\ & U & \end{tikzcd}$$

We also write V/U for an element $V \to U \in \mathcal{C}/U$. We can equip $\mathcal{C}/U$ with the structure of a site by declaring a family $(V_i \to V)_i$ of morphisms over U to be a covering iff it is a covering of V in $\mathcal{C}$. It is clear that this defines a site. As we required the existence of finite limits in sites, $\mathcal{C}$ has a terminal object X, for which we get an equivalence of categories $\mathcal{C} \cong \mathcal{C}/X$.
For any $U \in \mathcal{C}$ one can consider the functor

$$\begin{aligned} j_U^{-1} : \mathcal{C} &\longrightarrow \mathcal{C}/U \\ V &\longmapsto (V \times_X U \to U). \end{aligned}$$

It is an exercise to check that j_U^{-1} defines a morphism of sites $j_U : \mathcal{C}/U \to \mathcal{C}$.

Definition 1.1.25. Lef $\mathcal{F}$ be a sheaf on $\mathcal{C}$ and let $U \in \mathcal{C}$. Define the *restriction of $\mathcal{F}$ to U* by

$$\mathcal{F}|_U := j_U^* \mathcal{F}.$$

Note that the functor $j_U^* : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}/U)$ is exact and has a right adjoint j_{U*}.

Lemma 1.1.26. Lef $\mathcal{F}$ be a sheaf on $\mathcal{C}$ and let $U \in \mathcal{C}$. Then the restriction of $\mathcal{F}$ to U is given by

$$\mathcal{F}|_U(V/U) = \mathcal{F}(V)$$

for all $V/U \in \mathcal{C}/U$.

Proof. Let $j_U : \mathcal{C}/U \to \mathcal{C}$ be the morphism of sites which defines the restriction. To show the formula, we have to prove that the rule

$$\begin{aligned} \xi : \mathrm{Sh}(\mathcal{C}) &\longrightarrow \mathrm{Sh}(\mathcal{C}/U) \\ \mathcal{F} &\longmapsto (V/U \mapsto \mathcal{F}(V)) \end{aligned}$$

is the left adjoint of j_{U*}. First, it is clear that $\xi(\mathcal{F})$ is a sheaf and that ξ defines a functor. Let $f : \mathcal{F} \to j_{U*}\mathcal{G}$ be a morphism of sheaves, where $\mathcal{G} \in \mathrm{Sh}(\mathcal{C})$ and $\mathcal{F} \in \mathrm{Sh}(\mathcal{C}/U)$. Then we get a morphism $\xi(\mathcal{F}) \to \mathcal{G}$ defined on sections by

$$\xi(\mathcal{F})(V/U) = \mathcal{F}(V) \xrightarrow{f(V)} j_{U*}\mathcal{G}(V) \cong \mathcal{G}(V/U).$$

Conversely, any morphism of sheaves $g : \xi(\mathcal{F}) \to \mathcal{G}$ defines a morphism $\mathcal{F} \to j_{U*}\mathcal{G}$ as follows

$$\mathcal{F}(V) \to \mathcal{F}(V \times_X U) = \xi(\mathcal{F})(V \times_X U/U) \xrightarrow{g(V\times_X U)} \mathcal{G}(V \times_X U/U) = j_{U*}\mathcal{G}(V).$$

It is left to the reader to show that these two constructions are functorial in $\mathcal{F}$ and $\mathcal{G}$ and that they are mutually inverse. □

Lemma 1.1.27. Let $\epsilon : \mathcal{C} \to \mathcal{D}$ be a morphism of sites and $U \in \mathcal{D}$. Then ϵ induces a morphism of sites $\epsilon' : \mathcal{C}/\epsilon^{-1}(U) \to \mathcal{D}/U$ making the square

$$\begin{tikzcd} \mathcal{C}/\epsilon^{-1}(U) \arrow[r, "\epsilon'"] \arrow[d] & \mathcal{D}/U \arrow[d] \\ \mathcal{C} \arrow[r, "\epsilon"] & \mathcal{D} \end{tikzcd}$$

commute. Moreover, for a sheaf $\mathcal{F}$ on $\mathcal{D}$ we have the formula $(\epsilon^*\mathcal{F})|_{\epsilon^{-1}(U)} = \epsilon'^*(\mathcal{F}|_U)$. If it is clear from the context, we will write ϵ instead of ϵ'.

Proof. The morphism of sites ϵ' is given by the functor

$$\begin{aligned} \epsilon'^{-1} : \mathcal{D}/U &\longrightarrow \mathcal{C}/\epsilon^{-1}(U) \\ V/U &\longmapsto \epsilon^{-1}(V)/\epsilon^{-1}(U). \end{aligned}$$

It is clear that this defines a morphism of sites and makes the above square commute. The statement $(\epsilon^*\mathcal{F})|_{\epsilon^{-1}(U)} = \epsilon'^*(\mathcal{F}|_U)$ follows from Lemma 1.1.15. □

We will introduce the notion of locally constant sheaves for arbitrary sites and state some basic results, which also can be found in [15, Tag 093P]. Fix a site $(\mathcal{C}, \mathrm{Cov})$ and a ring Λ.

Definition 1.1.28. A sheaf F is *locally constant* if for every object $U \in \mathcal{C}$ there is a covering $(\varphi_i : U_i \to U)_{i\in I}$ such that $F|_{U_i}$ is a constant sheaf on $\mathcal{C}/U_i$.
We say a sheaf F of abelian groups or sets is *finite locally constant* if it is locally constant and the values of the constant sheaves $F|_{U_i}$ are finite abelian groups or sets.

Lemma 1.1.29. Assume Λ is a noetherian ring. The category of locally constant abelian sheaves is a weak Serre subcategory[4] of the category of abelian sheaves. Similarly, the locally constant sheaves of finite type Λ-modules form a weak Serre subcategory of the sheaves of finte type.

Proof. [15, Tag 093U] □

Lemma 1.1.30. Let $\mathcal{C}$ be a site. Finite limits and colimits of locally constant sheaves on $\mathcal{C}$ are again locally constant.

Proof. This is a direct consequence of the foregoing lemma. □

[4]See Section 1.2

1.1.4 Tensor Product and Internal Homomorphism

Let Λ be a ring. Two important constructions concerning sheaves of Λ-modules are the tensor product and its right adjoint, the internal homomorphism. The goal of this technical subsection is Proposition 1.1.36. We want the tensor product to commute with pullbacks along arbitrary morphisms of sites $\epsilon : \mathcal{C} \to \mathcal{D}$ to ensure that for $I \subset \Lambda$ an ideal and $\mathcal{F}$ a sheaf of Λ-modules on $\mathcal{D}$ the equality

$$\epsilon^*(I\mathcal{F}) \cong I\epsilon^*\mathcal{F}$$

holds.
For the following fix a site $\mathcal{C}$ and a sheaf of rings $\mathcal{O}$ on $\mathcal{C}$. We will later apply the theory to $\mathcal{O} = \underline{\Lambda}$ for a ring Λ.

Definition 1.1.31 (Tensor)**.** Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of $\mathcal{O}$-modules. Define the presheaf $\mathcal{F} \otimes_{\mathcal{O},\mathrm{PSh}} \mathcal{G}$ by the rule

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U).$$

Further set $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} := \mathrm{a}(\mathcal{F} \otimes_{\mathcal{O},\mathrm{PSh}} \mathcal{G})$.

Lemma 1.1.32. Tensoring commutes with sheafification, that is

$$\mathrm{a}(\mathcal{F} \otimes_{\mathcal{O},\mathrm{PSh}} \mathcal{G}) = \mathrm{a}\mathcal{F} \otimes_{\mathrm{a}\mathcal{O}} \mathrm{a}\mathcal{G}.$$

for a presheaf of rings $\mathcal{O}$ and presheaves of $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$.

Proof. [15, Tag 0GMW] □

Remark 1.1.33. Let Λ be a ring and denote by $\underline{\Lambda}$ the constant sheaf associated to Λ. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of Λ-modules. Then we often write

$$\mathcal{F} \otimes_{\Lambda} \mathcal{G} := \mathcal{F} \otimes_{\underline{\Lambda}} \mathcal{G}.$$

This is by Lemma 1.1.32 the same as the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\Lambda} \mathcal{G}(U).$$

Definition 1.1.34 (Internal Hom)**.** Let $\mathcal{F}$ be a presheaf and $\mathcal{G}$ a sheaf of $\mathcal{O}$-modules. The assertion

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}|_U, \mathcal{G}|_U)$$

defines a sheaf of $\mathcal{O}$-modules, see [15, Tag 04TT]. This sheaf is called the *internal homomorphism* and is denoted by $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

Proposition 1.1.35. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then the functor $\mathcal{F} \otimes_{\mathcal{O}} -$ is the left adjoint of the functor $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, -)$. In particular, for any sheaves of $\mathcal{O}$-modules $\mathcal{G}$ and $\mathcal{H}$ there is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}))$$

functorial in $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$.

Proof. [15, Tag 04TT] □

Proposition 1.1.36. Let $\epsilon : (\mathcal{C}, \mathrm{Cov}) \longrightarrow (\mathcal{C}', \mathrm{Cov}')$ be a morphism of sites. Further let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of $\mathcal{O}$ modules on $\mathcal{C}'$. Then

$$\epsilon^*(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}) = \epsilon^*\mathcal{F} \otimes_{\epsilon^*\mathcal{O}} \epsilon^*\mathcal{G}$$

Proof. [15, Tag 03EL] □

Definition 1.1.37. Let $\mathcal{O}$ be a sheaf of rings and $\mathcal{I} \subset \mathcal{O}$ a sheaf of ideals. For a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ set

$$\mathcal{I}\mathcal{F} := \mathrm{im}(\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}).$$

This is the same as the sheafification of the presheaf defined by

$$U \longmapsto \mathcal{I}(U)\mathcal{F}(U).$$

We will usually apply this to a ring Λ with an ideal $I \subset \Lambda$ and write $I\mathcal{F} := \underline{I}\mathcal{F}$.

Corollary 1.1.38. Let $\epsilon : \mathcal{C} \to \mathcal{D}$ be a morphism of sites. Let $I \subset \Lambda$ be an ideal and $\mathcal{F}$ a sheaf of Λ-modules on $\mathcal{D}$. We then have the equality

$$\epsilon^*(I\mathcal{F}) \cong I\epsilon^*\mathcal{F}.$$

Proof. Since ϵ^* is exact, it commutes with the image functor. Hence,

$$\epsilon^*(I\mathcal{F}) = \mathrm{im}(\epsilon^*(\underline{I} \otimes_{\Lambda} \mathcal{F}) \to \epsilon^*\mathcal{F}).$$

But by Lemma 1.1.19 and Proposition 1.1.36 this is the same as

$$\mathrm{im}(\underline{I} \otimes_{\Lambda} \epsilon^*\mathcal{F} \to \epsilon^*\mathcal{F}).$$

□

Lemma 1.1.39. If $\mathcal{F}$ and $\mathcal{G}$ are two locally constant sheaves of Λ-modules on $\mathcal{C}$ then

$$\mathcal{F} \otimes_{\Lambda} \mathcal{G}$$

is a locally constant sheaf of Λ-modules.

Proof. [15, Tag 093V] □

1.2 Interlude on Serre Subcategories

The material of this section can be found in [15, Tag 02MN].

Definition 1.2.1. Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subset \mathcal{A}$ a full subcategory. We say $\mathcal{C}$ is a *Serre subcategory* of $\mathcal{A}$ if for every exact sequence

$$A \longrightarrow B \longrightarrow C$$

with $A, C \in \mathcal{C}$ we already have $B \in \mathcal{C}$.
Similarly, $\mathcal{C}$ is a *weak Serre subcategory* if for any exact sequence

$$A_1 \to A_2 \to B \to C_1 \to C_2$$

with $A_1, A_2, C_1, C_2 \in \mathcal{C}$ we already have $B \in \mathcal{C}$.

Serre subcategories have useful properties, which we will use later in this thesis. In the following we state two of them.

Lemma 1.2.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subset \mathcal{A}$ a Serre subcategory. Then any quotient and any subobject of an element in $\mathcal{C}$ is again in $\mathcal{C}$.

Proof. [15, Tag 02MP] □

The main advantage of Serre subcategories is that we can define quotients of categories.

Proposition 1.2.3. Let $\mathcal{A}$ be an abelian category with a Serre subcategory $\mathcal{C}$. Then there exists an abelian category $\mathcal{A}/\mathcal{C}$ together with an essential surjective exact functor

$$F : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

which has kernel $\mathcal{C}$, i.e. the category of elements in $\mathcal{A}$, which are mapped to the zero object via F is exactly $\mathcal{C}$. Moreover, F and $\mathcal{A}/\mathcal{C}$ are characterized by the following universal property. Any exact functor $G : \mathcal{A} \to \mathcal{B}$ in a category $\mathcal{B}$ with $\mathcal{C} \subset \ker(G)$ factorizes as $G = H \circ F$ for a unique exact functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$.

Proof. [15, Tag 02MS] □

Remark 1.2.4. The foregoing proposition shows that a morphism $f : A \to B$ becomes an isomorphism after applying F if and only if the kernel and cokernel of f are elements in $\mathcal{C}$.

Lemma 1.2.5. Let $\mathcal{A}$ be an abelian category. A subcategory $\mathcal{S} \subset \mathcal{A}$ is a Serre subcategory of $\mathcal{A}$ if and only if the following conditions are fulfilled

- The zero element lies in $\mathcal{S}$.

- $\mathcal{S}$ is a strictly full subcategory of $\mathcal{A}$.
- $\mathcal{S}$ is closed under kernels and cokernels.
- For any short exact sequence

$$0 \to F \to G \to H \to 0$$

  with $F, H \in \mathcal{S}$, we already have $G \in \mathcal{S}$.

Proof. This can be easily shown using the definitions. The proof is left to the reader. □

1.3 Completion of Modules

In this section we will recall some facts about the completion of modules. Note that we will leave out many statements and only concentrate on the parts that are interesting for this thesis. For more detailed material one can consult the respective section in the Stacks Project [15, Tag 00M9].
Let R be a ring and $I \subset R$ an ideal.

Definition 1.3.1. The *completion of R with respect to I* is the algebra

$$R^\wedge := \lim_n R/I^n R.$$

For an R-module M define the *completion of M* as

$$M^\wedge := \lim_n M/I^n M$$

This definition directly gives a canonical map $M \to M^\wedge$ and for a morphism of R-modules $M \to N$ there is an induced map $M^\wedge \to N^\wedge$ such that

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ M^\wedge \arrow[r] & N^\wedge \end{tikzcd}$$

commutes.

Lemma 1.3.2 (basic properties)**.**

1. If $M \to N$ is surjective then the induced map $M^\wedge \to N^\wedge$ is surjective.
2. If $M/IM \to N/IN$ is surjective then $M^\wedge \to N^\wedge$ is surjective.
3. The completion commutes with finite direct sums.

Proof. See [15, Tag 0315] for 1 and 2. The third can be proven using a straightforward calculation and the fact that a finite direct sum is the same as a finite product in the category of R-modules. □

Definition 1.3.3. An R-module M is called *(I-adically) complete* if the canonical map

$$M \to M^\wedge$$

is an isomorphism. The ring R is called *complete* if R is complete as R-module.

Remark 1.3.4. The completion of modules over arbitrary rings is in general not complete, for instance see [15, Tag 05JA]. However, this is true if I is a finitely generated ideal, so we will usually restrict to this case.

Lemma 1.3.5. Assume I is finitely generated as an ideal and let M be an R-module. Then

- $M^\wedge$ is I-adically complete.
- The map $M \to M^\wedge$ induces an isomorphism $M^\wedge/I^n M^\wedge \cong M/I^n M$. In particular $I^n M^\wedge = (I^n M)^\wedge = \ker(M^\wedge \to M/I^n M)$.

Proof. The first assertion directly follows from the second taking limits,

$$M^\wedge = \lim_n M/I^n M \cong \lim_n M^\wedge/I^n M^\wedge = (M^\wedge)^\wedge.$$

It remains to prove the first part. Since I is finitely generated, I^n is finitely generated. Let $I^n = (f_1, \dots, f_n)$ with $f_i \in I$. Now consider the surjective map

$$(f_1, \dots, f_n) : \bigoplus_{i=1}^{n} M \to I^n M$$

induced by multiplying the i'th component by f_i. Property 1 in Lemma 1.3.2 gives a surjection $\tau : \bigoplus_{i=1}^n M^\wedge \to (I^n M)^\wedge = \ker(M^\wedge \to M/I^n M)$. But the image of τ in $M^\wedge$ is exactly $I^n M^\wedge$, which proves

$$\ker(M^\wedge \to M/I^n M) = I^n M^\wedge.$$

This proves the claim since the canonical projections $M^\wedge \to M/I^n M$ are surjective. $\square$

Lemma 1.3.6. A direct summand of a complete R-module is complete.

Proof. Let $M = M_1 \oplus M_2$ be a complete R-module. We use Lemma 1.3.2 and the construction of the completion to get a commutative diagram with surjections and injections as indicated

$$\begin{tikzcd} M_1 \arrow[r, hook] \arrow[d, "\tau"'] & M = M_1 \oplus M_2 \arrow[r, two heads] \arrow[d, "\cong"] & M_1 \arrow[d, "\tau"] \\ M_1^\wedge \arrow[r] & M^\wedge \arrow[r, two heads] & M_1^\wedge, \end{tikzcd}$$

where τ is the canonical map. The right square implies that τ is surjective and the left square implies that τ is injective. Therefore, τ is an isomorphism. $\square$

Our main result of this section is the following.

Proposition 1.3.7 (Recovering inverse systems)**.** Assume I is finitely generated and let $(M_n)_{n\in\mathbb{N}}$ be an inverse system of R-modules, i.e. a chain

$$M_0 \leftarrow \ldots \leftarrow M_n \leftarrow \ldots$$

such that $I^{n+1}M_n = 0$. Then $M := \lim_n M_n$ is I-adically complete. If further the induced maps $M_{n+1}/I^{n+1}M_{n+1} \to M_n$ are isomorphisms then we can recover the inverse system from M via isomorphisms

$$M/I^{n+1}M \cong M_n.$$

If additionally all M_n are finite R-modules, then M is a finite $R^\wedge$-module.

[15, Tag 09B8]. The canonical projection $M \to M_n$ factors as $M \to M/I^{n+1}M \to M_n$. Taking the limit over n yields to

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & M^\wedge \arrow[r] & M. \end{tikzcd}$$

But this implies that $M^\wedge$ fits in a splitting short exact sequence

$$0 \to N \to M^\wedge \to M \to 0$$

which implies that M is a direct summand of the I-adically complete R-module $M^\wedge$ (see Lemma 1.3.5). Then M is complete by Lemma 1.3.6.
Now additionally assume $M_{n+1}/I^{n+1}M_{n+1} \cong M_n$. Let N_n be the kernel of the surjective projections $\tau_n : M \to M_n$. Consider the commutative triangle

$$\begin{tikzcd} M \arrow[r, "\tau_{n+1}"] \arrow[dr, "\tau_n"'] & M_{n+1} \arrow[d] \\ & M_n. \end{tikzcd}$$

We easily detect that $N_n = \ker(\tau_n) = \ker(\tau_{n+1}) + I^{n+1}M = N_{n+1} + I^{n+1}M$. Hence we get well-defined surjections

$$N_{n+1}/I^{n+2}M \twoheadrightarrow N_n/I^{n+1}M.$$

Keeping this result in mind, we consider the short exact sequence

$$0 \to N_n/I^{n+1}M \to M/I^{n+1}M \to M_n \to 0.$$

Applying the inverse limit is left exact. Hence, we obtain a left exact sequence

$$0 \to \lim_n N_n/I^{n+1}M \to M^\wedge \to M.$$

Since M is complete, the right arrow is an isomorphism and $0 = \lim_n N_n/I^{n+1}M$ must hold. But the result from above indicates the surjectivity of the transition maps $N_{n+1}/I^{n+2}M \twoheadrightarrow N_n/I^{n+1}M$. We deduce that $0 = N_n/I^{n+1}M$, which finally implies $M/I^{n+1}M \cong M_n$.
Now assume that all M_n are finite R-modules. In particular, the quotient M/IM is a finite R/I-module and hence there is a surjection

$$(R/I)^n = R^n/IR^n \longrightarrow M/I$$

By Lemma 1.3.2 we get a surjection

$$(R^n)^\wedge = (R^\wedge)^n \to M^\wedge,$$

which completes the proof. □

Proposition 1.3.8. If M is a finite Λ-module and Λ a noetherian ring then

$$M^\wedge \cong M \otimes_\Lambda \Lambda^\wedge$$

In particular, every finite module over a complete noetherian ring is complete.

Proof. [15, Tag 00MA] □

Lemma 1.3.9. Let $I = (f_1, \ldots, f_l) \subset R$ be a finitely generated ideal and M an R-module. Then

$$M^\wedge = \lim_n M/(f_1^n, \ldots, f_l^n)M$$

Proof. This is an easy computation using the inclusions $I^{rn} \subset (f_1^n, \ldots, f_l^n) \subset I^n$. □

Lemma 1.3.10. Let $I = (a_1, \ldots, a_n) \subset R$ be a finitely generated ideal and M an R-module. If the canonical maps

$$M \to \lim_n M/a_i^n M$$

are surjective for all i, then the map $M \to M^\wedge$ into the I-adic completion of M is surjective.

Proof. ([15, Tag 090S]) By Lemma 1.3.9 we have $M^\wedge = \lim_n M/(f_1^n, \ldots, f_l^n)M$. Hence, an element $x \in M^\wedge$ can be represented by a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in M/(f_1^n, \ldots, f_l^n)M$ and $x_{n+1} - x_n \in (f_1^n, \ldots, f_l^n)M$. We write

$$x_{n+1} - x_n := \sum_{i=1}^{l} a_{n,i} f_i^n$$

with $a_{n,i} \in M$. For $i \in \{1, \ldots, l\}$ choose preimages $y_i \in M$ for the elements

$$\left(\left[\sum_{j=1}^{n} a_{j,i} f_i^j \right] \right)_{n\in\mathbb{N}} \in \lim_n M/(f_i^n)M.$$

Then a preimage of x is given by $y := \sum_{i=1}^{l} y_i$. □

Lemma 1.3.11. Let $I = (a_1, \ldots, a_r) \subset R$ be a finitely generated ideal and M an R-module. If M is (a_i)-adically complete for all i, then M is I-adically complete.

Proof. We use induction on r. For $r = 1$ the claim is trivial.
Assume $r > 1$. The canonial morphism $M \to M^\wedge$ is surjective by Lemma 1.3.10. Let $x \in M$ with $x \in \bigcap_{n\in\mathbb{N}} I^n M = \ker(M \to M^\wedge)$. As $I^{rn} \subset (a_1^n, \ldots, a_r^n)$, we see that $x \in \bigcap_{n\in\mathbb{N}} (a_1^n, \ldots, a_r^n)$. This exactly means that for any $n \in \mathbb{N}$ there are $y_{1,n}, \ldots, y_{r,n} \in M$ such that

$$x = a_1^n y_{1,n} + \ldots + a_r^n y_{r,n}.$$

Define the ideal $I' := (a_2, \ldots, a_r)$. The canonical map $\pi : M \to \lim_n M/I'M$ is an isomorphism by the induction hypothesis. But for every $m \in \mathbb{N}$ we can represent $\pi(x)$ by the sequence

$$([a_1^n y_{1,n}])_{n\in\mathbb{N}} = a_1^m ([a_1^n y_{1,n+m}])_{n\in\mathbb{N}}.$$

This shows that $x \in \bigcap_{n\in\mathbb{N}} (a_1^n) = \{0\}$. Therefore, $M \to M^\wedge$ is injective which completes the proof. □

2 The Étale Site

In this chapter we want to introduce the étale site and provide some initial properties. Most of the statements are standard and can be found in nearly any textbook on étale cohomology. However, we mainly used [6] as reference for this chapter.

2.1 Definitions and Basic Statements

From now on, fix a scheme X.

Definition 2.1.1. The étale site consists of the category of étale schemes over X together with the coverings given by

$$\mathrm{Cov}_{\mathrm{et}}(U) := \left\{ (f_i : U_i \to U)_{i \in I} \mid \text{all } f_i \text{ are étale and } U = \bigcup_{i \in I} f(U_i) \right\}.$$

Isomorphisms are well-known to be étale morphisms, and the property of being étale is preserved under composition and base change. It is trivial to verify the definitions of a site. We denote this site by X_{et} and call sheaves on X_{et} *étale sheaves.*

Remark 2.1.2. Let $f : X \to Y$ be a morphism of schemes. Then we get a canonical morphism of sites $f : X_{\mathrm{et}} \to Y_{\mathrm{et}}$ defined by the functor

$$\begin{aligned} f^{-1} : Y_{\mathrm{et}} &\longrightarrow X_{\mathrm{et}} \\ (V/Y) &\longmapsto (f^{-1}(V) := V \times_Y X/X). \end{aligned}$$

Note that for any étale morphism $g : U \to X$ the sites U_{et} and X_{et}/U coincide and that the morphism of sites $U_{\mathrm{et}} \to X_{\mathrm{et}}$ is exactly the morphism of sites defined by the restriction from Section 1.1.3. If $\iota : Z \hookrightarrow X$ is a locally closed subscheme, define the restriction of a sheaf F as $F|_Z := \iota_{\mathrm{et}}^* F$.

Remark 2.1.3. The assertion $X \mapsto X_{\mathrm{et}}$ defines a functor

$$(-)_{\mathrm{et}} : \mathrm{Sch} \longrightarrow \mathrm{Sites}$$

from the category of schemes into the category of sites. Of course, one has to be careful as it is a functor between 2-categories. For two composable morphisms of schemes, f and g, the associated morphism of sites $(f \circ g)_{\mathrm{et}}$ might differ from the composition $f_{\mathrm{et}} \circ g_{\mathrm{et}}$. However, there is always a canonical natural isomorphism between $f_{\mathrm{et}} \circ g_{\mathrm{et}}$ and $(f \circ g)_{\mathrm{et}}$. We will ignore such types of problems as they do not cause any issues in this thesis.

Examples 2.1.4.

- Let A be a non-trivial abelian group. We will see that the constant étale sheaf associated to A is non-trivial.
- Let X be a scheme. For any étale morphism $U \to X$ and any $N \in \mathbb{N}$ define

  $$\mu_{X,N}(U) := \{f \in \mathcal{O}_U(U)^\times \mid f^N = 1\}.$$

  This is a sheaf, called the N'th roots of unity.

Stalks of Étale Sheaves

One of the main advantages in the study of sheaves on the Zariski site are stalks. Many statements can be reduced to a stalkwise calculation, which is often easier. This concept generalizes, in a certain sense, to sheaves on the étale site. We will shortly explain the construction of étale stalks, and provide an overview over the most important properties. For a more detailed insight see [6, Section 5.3] or [16, §5].

Definition 2.1.5. Let $x \in X$ be any point and let G be a presheaf with values in Set or Ab on X_{et}. A *geometric point* centered in x is a morphism $s : \mathrm{Spec}(\Omega) \to X$ which factorizes through $\mathrm{Spec}(\kappa(x)) \to X$ and where Ω is a separably closed field. We also write $s := \mathrm{Spec}(\Omega)$ or $s \to X$. For such a geometric point s we define the *stalk of G at s* as

$$G_s := \Gamma(\mathrm{Spec}(\Omega), s^\bullet G)$$

Definition 2.1.6. Let $s : \mathrm{Spec}(\Omega) \to X$ be a geometric point. An *étale neighborhood* of s is a commutative diagram

$$\begin{tikzcd} & U \arrow[d] \\ \mathrm{Spec}(\Omega) \arrow[ur, "u"] \arrow[r, "s"'] & X \end{tikzcd}$$

where $U \to X$ is an étale map. A morphism of étale neighborhoods $(U,u) \to (V,v)$ is a morphism $\varphi : U \to V$ over X such that $\varphi \circ u = v$. Note that the category I_s of étale neighborhoods of s is a cofiltered category.

Proposition 2.1.7. If G is an étale presheaf and s a geometric point, then the stalk can be computed as

$$G_s \cong \mathrm{colim}_{(U,u) \in I_s} G(U).$$

Moreover, sheafification preserves stalks. That is $aG_s \cong G_s$.

Proof. [15, Tag 03PT] □

Example 2.1.8. Let A be an abelian group and $\underline{A}$ the sheaf associated to the constant presheaf with values in A. Then for any geometric point s we have $\underline{A}_s \cong A$. This follows directly from Proposition 2.1.7. In particular, $\underline{A}$ is an example of a non-trivial sheaf on the étale site.

Proposition 2.1.9. Let $s : \mathrm{Spec}(\Omega) \to X$ be a geometric point in X and $F \in \mathrm{Sh}(X_{\mathrm{et}})$ with values in $\mathcal{A} = \mathrm{Set}$ or Ab. Then

1. The functor given by $F \mapsto F_s$ is exact.
2. For any morphism of geometric points[1] $s \to s'$ we have a canonical isomorphism
$$F_s \cong F_{s'}$$
3. A morphism of schemes $f : Y \to X$ induces an isomorphism
$$(f^*F)_t \cong F_{f(t)}$$
for any geometric point t of Y. Here $f(t) := f \circ t$.

Proof. The details can be found in [6, Section 5.3]. □

Definition 2.1.10. For every $x \in X$ fix a separable closure $\overline{\kappa(x)}$ of $\kappa(x)$. Then define $\overline{x} := \mathrm{Spec}(\overline{\kappa(x)})$. We denote the corresponding geometric point by $\overline{x}$.

Theorem 2.1.11. Let F and G be sheaves on X_{et} and $\varphi : F \to G$ a morphism of sheaves. Then φ is an isomorphism, an epimorphism, an monomorphism or the zero morphism if and only if for every $x \in X$ the morphism on stalks $\varphi_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}}$ is so. Further the canonical map
$$F(X) \to \prod_{x \in X} F_{\overline{x}}$$
is injective and a sequence of abelian sheaves
$$F \to G \to H$$
is exact if and only if for every $x \in X$ the sequence
$$F_{\overline{x}} \to G_{\overline{x}} \to H_{\overline{x}}$$
is exact.

Proof. [6, Prop. 5.3.3] □

Proposition 2.1.12. Let $f : Y \to X$ be an immersion of schemes and $F \in \mathrm{Sh}(Y_{\mathrm{et}})$. Then the canonical morphism
$$f^* f_* F \to F$$
is an isomorphism.

Proof. This follows from a stalkwise calculation. For instance, a proof can be found in [6, Cor. 5.3.8]. □

[1] A morphism of geometric points is a morphism of schemes over X

2.2 Locally Constant Sheaves on the Étale Site

An important concept which allows to detect locally constant sheaves on the étale site is the specialization. Let X be a scheme and $s : \mathrm{Spec}(\Omega) \to X$ a geometric point with image $x \in X$. Then define $\widetilde{\mathcal{O}}_{X,s}$ to be the strict henselization of the local ring $\mathcal{O}_{X,x}$. Further set $\widetilde{X}_s := \mathrm{Spec}(\widetilde{\mathcal{O}}_{X,s})$. Note that the map $s : \mathrm{Spec}(\Omega) \to X$ factorizes as $\mathrm{Spec}(\Omega) \to \widetilde{X}_s \to X$ where $\widetilde{X}_s \to X$ is the canonical map.
Assume s, s' are two geometric points in X. A specialization morphism is an X-morphism $\widetilde{X}_{s'} \to \widetilde{X}_s$. If such a morphism exists we say s is a *specialization* of s'.

Lemma 2.2.1. A geometric point s is the specialization of a geometric point s' if and only if $x \in \overline{\{x'\}}$ where x (resp. x') is the image of s (resp. s') in X.

Proof. [6, Prop. 5.3.5] □

Lemma 2.2.2. Let F be a sheaf on X_{et} and s, s' two geometric points. A specialization morphism $\widetilde{X}_{s'} \to \widetilde{X}_s$ induces a canonical morphism of sheaves

$$F_s \to F_{s'}.$$

We will call this morphism the *specialization map* associated to $\widetilde{X}_{s'} \to \widetilde{X}_s$.

Proof. Assume $s : \mathrm{Spec}(\Omega) \to X$ and $s' : \mathrm{Spec}(\Omega') \to X$ are two geometric points. Let (U, u) be an étale neighborhood of s. Then $u : \mathrm{Spec}(\Omega) \to U$ induces a map $\widetilde{X}_s \to U$. Composing this map with the given specialization morphism yields to a morphism

$$v_U : \mathrm{Spec}(\Omega') \to \widetilde{X}_{s'} \to \widetilde{X}_s \to U.$$

One easily sees that this defines a geometric neighborhood of s'. To check that this construction is functorial in the geometric neighborhood of s is left to the reader. Hence the identity $F(U) \to F(U)$ induces a map

$$F_s \cong \mathrm{colim}_{(U,u)\in I_s} F(U) \longrightarrow \mathrm{colim}_{(V,v)\in I_{s'}} F(V) \cong F_{s'}.$$

□

Proposition 2.2.3. Let X be a noetherian scheme and R a noetherian ring. A sheaf F with finite stalks (resp. finitely generated stalks) is locally constant if and only if for any specialization morphism $f : \widetilde{X}_{s'} \to \widetilde{X}_s$ the specialization map $F_s \to F_{s'}$ associated to f is an isomorphism.

Proof. [6, Prop. 5.8.9] □

2.3 Constructible Sheaves on the Étale Site

Definition 2.3.1. Let X be a topological space. A partition of X is a disjoint decomposition of X in locally closed subsets[2]. That is $X = \bigsqcup_{i \in I} X_i$ with $X_i \subset X$ locally closed. A *constructible* subset of X is a subset $E \subset X$ such that E is the finite union of subsets of the form $U \cap (X \setminus V)$, where U and V are retrocompact open subsets of X. Recall that a subset is called *retrocompact* if its intersection with any compact subset of X is compact. A partition $X = \bigsqcup_{i \in I} X_i$ is called *stratification* if the X_i are locally closed and constructible[3].

Definition 2.3.2. Let X be a scheme and Λ a ring.

1. Let F be a sheaf of abelian groups on X_{et}. We say F is *constructible* if for every affine open $U \subset X$ there is a finite partition $U = \bigsqcup_{i=1}^n U_i$ in constructible locally closed subsets of U such that $F|_{U_i}$ is finite locally constant.
2. Let F be a sheaf of Λ-modules on X_{et}. We say F is *constructible* if for every affine open $U \subset X$ there is a finite partition $U = \bigsqcup_{i=1}^n U_i$ in constructible locally closed subsets of U such that $F|_{U_i}$ is locally constant and of finite type.

Remark 2.3.3. Let Λ be a finite ring and consider a constructible sheaf of Λ-modules F. Obviously, F is a constructible abelian sheaf. This is in general not correct if the considered ring is not finite. One reason why the definition remains valid is that for X quasi-separated and quasi-compact the constructible abelian sheaves (resp. the constructible sheaves of Λ-modules) form the smallest subcategory of $\mathrm{Sh}(X_{\text{et,Ab}})$ (resp. of $\mathrm{Mod}(\Lambda)$) which is closed under limits and contains sheaves of the form $j_{U!}\mathbb{Z}/n\mathbb{Z}$ (resp. $j_{U!}\underline{\Lambda}$). Here $j_{U!}$ denotes the extension by zero[4] and U is any quasi-compact and quasi-separated étale scheme over X.

Proposition 2.3.4. If $f : X \to Y$ is a morphism of noetherian schemes then the pullback of a constructible sheaf along f is again constructible.

Proof. [6, Proposition 5.8.2] □

Proposition 2.3.5. Let X be a scheme and Λ a noetherian ring. For two constructible sheaves of Λ-modules $\mathcal{F}$ and $\mathcal{G}$ the tensor product $\mathcal{F} \otimes_\Lambda \mathcal{G}$ is constructible.

Proof. Let $U \subset X$ be an affine open. Pick partitions $U = \bigsqcup_{i=1}^n U_i$ and $U = \bigsqcup_{j=1}^m V_j$ which arise from the constructibility of $\mathcal{F}$ and $\mathcal{G}$. Then $U = \bigsqcup_{i,j} U_i \cap V_j$ is a partition into constructible locally closed subsets such that

$$(\mathcal{F} \otimes_\Lambda \mathcal{G})|_{U_i \cap V_j} = \mathcal{F}|_{U_i \cap V_j} \otimes_\Lambda \mathcal{G}|_{U_i \cap V_j}$$

is locally constant. □

[2] I.e. subsets which are of the form $C \overset{i}{\hookrightarrow} U \overset{j}{\hookrightarrow} X$ such that i is a closed and j an open immersion.
[3] This may be non-standard notation.
[4] See Section 4.2.

Lemma 2.3.6. The category of abelian constructible sheaves is a weak Serre subcategory of the category of abelian sheaves on X_{et}. The same is true for the constructible sheaves of Λ-modules inside the category $\mathrm{Mod}_\Lambda(X_{\mathrm{et}})$. In particular, the category of constructible sheaves is closed under finite limits and colimits.

Proof. [15, Tag 03RZ] □

Proposition 2.3.7. Let R be a noetherian ring and X a noetherian scheme. Then a sheaf of R-modules F on X_{et} is constructible if and only if F is noetherian. That is, any ascending chain

$$F_0 \subset F_1 \subset \ldots \subset F$$

of subsheaves of F gets eventually constant. In particular, this implies that any subsheaf and any quotient of a constructible sheaf of R-modules is again constructible.

Proof. [6, Prop. 5.8.6] □

Corollary 2.3.8. If R is a noetherian ring and X a noetherian scheme then the constructible sheaves of R-modules form a Serre subcategory of the category of sheaves of R-modules on X_{et}.

Proof. This is covered by the "in particular"-part of Proposition 2.3.7 combined with Lemma 2.3.6. □

3 l-adic Formalism

In this chapter we want to define I-adic sheaves on the étale site. To do so, it is useful to introduce the formalism that comes with the Artin-Rees category. Later, we will need this technical chapter in order to derive many interesting properties of pro-étale sheaves by reducing to the étale case. We use the ideas from [6, Chapter 10] and [5, §12]. In the literature the ring $R := \mathbb{Z}_\ell$ of ℓ-adic numbers is used. However, we demand R to be an arbitrary noetherian ring as the proofs that are relevant for this thesis can be adapted in a straightforward manner.

3.1 The Artin Rees Category

In the following let $\mathcal{A}$ be an abelian category. For two categories $\mathcal{B}$ and $\mathcal{C}$ let $\underline{\mathrm{Func}}(\mathcal{B},\mathcal{C})$ be the category of functors $\mathcal{B} \to \mathcal{C}$. Furthermore, let $\mathbb{N}$ be the category whose elements are natural numbers, with a unique morphism $m \to n$ whenever $m \leq n$.

Definition 3.1.1. The category $\underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}},\mathcal{A})$ is called the category of *inverse systems* in $\mathcal{A}$. A functor $\mathcal{F} \in \underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}},\mathcal{A})$ is given by a family $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of objects in $\mathcal{A}$ together with transition morphisms $\mathcal{F}_{n+1} \to \mathcal{F}_n$ for all $n \in \mathbb{N}$:

$$\mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \ldots \leftarrow \mathcal{F}_n \ldots$$

Define $\mathcal{F}[m]$ as the inverse systems with

$$\mathcal{F}[m]_n := F_{m+n}$$

and obvious transition maps. This defines a functor $[m] : \underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}},\mathcal{A}) \to \underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}},\mathcal{A})$. Further we have canonical maps $\mathcal{F}[m] \to \mathcal{F}$ for any $m \geq 0$.

Lemma 3.1.2. The category $\underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}},\mathcal{A})$ is abelian and the functor $[m]$ is exact.

Proof. This is easy to check. For instance, we give a description of kernels and cokernels. Let $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ be two projective systems and

$$f : (F_n)_{n\in\mathbb{N}} \to (G_n)_{n\in\mathbb{N}}$$

a morphism. Then $\ker(f)$ (resp. $\mathrm{coker}(f)$) is given by the system $(\ker(f_n))_{n\in\mathbb{N}}$ (resp. by $(\mathrm{coker}(f_n))_{n\in\mathbb{N}}$), where f_n denotes the morphism $f(n) : F_n \to G_n$. The transition maps are the canonical ones. □

Definition 3.1.3. Let $\mathcal{F} \in \underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A})$ be a projective system. We say $\mathcal{F}$ fulfills the *Mittag-Leffler condition (ML)* if for any integer n there is an $r \geq n$ such that for all $t \geq r$ the equality

$$\mathrm{im}(\mathcal{F}_t \to \mathcal{F}_n) = \mathrm{im}(\mathcal{F}_r \to \mathcal{F}_n)$$

is given. The projective system fulfills the *Artin-Rees-Mittag-Leffler condition (ARML)* if there is an $r \in \mathbb{N}$ such that for all $t \geq r$ we have

$$\mathrm{im}(\mathcal{F}[r] \to \mathcal{F}) = \mathrm{im}(\mathcal{F}[t] \to \mathcal{F}).$$

Definition 3.1.4. Define $\underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A})^0$ as the full subcategory of $\underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A})$ with objects given by the objects $\mathcal{F}$ which have the property that the canonical map

$$\mathcal{F}[n] \to \mathcal{F}$$

is the zero mapping for $n \gg 0$. Such an $\mathcal{F}$ is called *null system*.

Proposition 3.1.5. The subcategory $\underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A})^0 \subset \underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A})$ is a Serre subcategory.

Proof. Let

$$\mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H}$$

be an exact sequence of projective systems with $\mathcal{F}$ and $\mathcal{H}$ null systems. This implies the existence of an integer m such that $\mathcal{F}[m] \to \mathcal{F}$ and $\mathcal{H}[m] \to \mathcal{H}$ are the zero mappings. We claim that $\mathcal{G}[2m] \to \mathcal{G}$ is zero and hence $\mathcal{G}[r] \to \mathcal{G}$ is zero for $r \geq 2m$. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
\mathcal{F}[2m] \arrow[r, "\iota[2m]"] \arrow[d, "0"'] & \mathcal{G}[2m] \arrow[r, "\pi[2m]"] \arrow[d, "g'"] & \mathcal{H}[2m] \arrow[d, "0"] \\
\mathcal{F}[m] \arrow[r, "\iota[m]"] \arrow[d, "0"'] & \mathcal{G}[m] \arrow[r, "\pi[m]"] \arrow[d, "g"] & \mathcal{H}[m] \arrow[d, "0"] \\
\mathcal{F} \arrow[r, "\iota"] & \mathcal{G} \arrow[r, "\pi"] & \mathcal{H}.
\end{tikzcd}$$

We see that $\pi[m] \circ g' = 0$ and hence g' factors through $\ker(\pi[m]) = \mathrm{im}(\iota[m])$. But $g \circ \iota = 0$ and hence $\mathrm{im}(\iota[m]) \hookrightarrow \ker(g)$ which implies that $g \circ g' = 0$. This proves the claim. □

Definition 3.1.6. Define the *AR-category* of projective systems in $\mathcal{A}$ as the quotient category

$$\mathrm{AR}(\mathcal{A}) := \underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A}) / \underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A})^0$$

from Proposition 1.2.3. This is an abelian category in which null systems are zero elements.

Later, we will need to work explicitly with the AR-category. Therefore, it is useful to give an alternative definition of the AR-category to gain easier access to morphisms between its objects.

Lemma 3.1.7. We can describe $\mathrm{AR}(\mathcal{A})$ as the category with

$$\begin{aligned}&\text{Objects: Inverse systems } \mathcal{F} \in \underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A})\\&\text{Morphisms: } \mathrm{Hom}_{\mathrm{AR}}(\mathcal{F}, \mathcal{G}) := \mathrm{colim}_n \mathrm{Hom}(\mathcal{F}[n], \mathcal{G})\end{aligned}$$

Hence, a morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{AR}(\mathcal{A})$ is represented by a morphism $\tilde{f} : \mathcal{F}[r] \to \mathcal{G}$ in $\underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathcal{A})$ and f is an isomorphism in the AR-category if and only if $\ker(\tilde{f})$ and $\mathrm{coker}(\tilde{f})$ are null systems.

Proof. This is surely a well-known fact, and we will omit the details. For instance, one can consult [6, p.530]. □

Lemma 3.1.8. Let $\mathcal{A}$ be an abelian category which has all small limits. The assertion $(F_n)_{n\in\mathbb{N}} \mapsto \lim_n F_n$ defines a functor

$$\mathrm{AR}(\mathcal{A}) \to \mathcal{A}.$$

Proof. This follows from the fact that $\lim_n F_n = \lim_n F_{n+r}$ for any inverse system F and any $r \geq 0$. This means that the inverse limit sends F and $F[r]$ to the same object. In particular, applying the inverse limit to a morphism $F[r] \to G$ or to the composition $F[r+r'] \to F[r] \to G$ yields the same result. □

3.2 Adic Systems of Modules

In the following, let R be a noetherian ring and $I \subset R$ an ideal.

Definition 3.2.1. An inverse system $(M_n)_{n\in\mathbb{N}}$ of R-modules is called an *I-adic system* if each M_n is finitely generated and the following properties are fulfilled.

- $I^{n+1}M_n = 0$
- The maps $M_{n+1} \to M_n$ induce isomorphisms $M_{n+1}/I^{n+1}M_{n+1} \cong M_n$.

Now consider the AR-category of finitely generated R-modules. An object in this category is called *AR I-adic* if it is isomorphic in the AR-category to an I-adic system of R-modules.

Example 3.2.2. The reason for introducing the Artin-Rees category is that I-adic systems do not form an abelian subcategory of the inverse systems of R-modules. Indeed, the kernel of the map

$$\phi : (\mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\cdot p} (\mathbb{Z}/p^n\mathbb{Z})$$

is given by $(p^{n-1}\mathbb{Z}/p^n\mathbb{Z})$ with zero as transition maps. These are obviously not surjective which implies that $\ker(\phi)$ is not I-adic. This is a problem, as many constructions and statements rely on the usage of abelian categories. The solution, if R is noetherian, is the passage to the AR-category. We will see that $(\ker(f_n))_{n\in\mathbb{N}}$ is at least AR I-adic and moreover that the category of AR I-adic systems is indeed an abelian subcategory of $\underline{\mathrm{Func}}(\mathbb{N}^{\mathrm{opp}}, \mathrm{Mod}_R)$.

Lemma 3.2.3. Let $M = (M_n)_{n\in\mathbb{N}}$ be an I-adic system of R-modules and $N = (N_n)_{n\in\mathbb{N}}$ an inverse system of sheaves with $I^{n+1}N_n = 0$ for any $n \in \mathbb{N}$, then

$$\mathrm{Hom}(M,N) \cong \mathrm{Hom}_{\mathrm{AR}}(M,N).$$

In particular, the category of I-adic modules is in equivalence with the category of AR I-adic modules.

Proof. By Lemma 3.1.7 we see

$$\mathrm{Hom}_{\mathrm{AR}}(M,N) \cong \mathrm{colim}_r \mathrm{Hom}(M[r],N).$$

Hence, an element in $\mathrm{Hom}_{\mathrm{AR}}(M,N)$ comes from an element $f \in \mathrm{Hom}(M[r],N)$ for some $r \in \mathbb{N}$. The morphism f consists of morphisms $f_n : M_{n+r} \to N_n$. Moreover, for $r \geq 1$ we have $M_{n+r-1} \cong M_{n+r}/I^{n+r}M_{n+r}$ and $I^{n+r}N_n = 0$ by assumption. But this implies that f uniquely factorizes as $f : M[r] \to M[r-1] \to N$. Proceeding inductively one gets a factorization $f : M[r] \to M \xrightarrow{g} N$ for a unique morphism $g \in \mathrm{Hom}(M,N)$. Consequently, g represents the same element as f in $\mathrm{Hom}_{\mathrm{AR}}(M,N)$ and g is the only element with this property. □

Proposition 3.2.4. Assume R is complete. The functor

$$\begin{aligned}\phi : \{\text{AR } I\text{-adic systems of } R\text{-modules}\} &\longrightarrow \{\text{finite } R\text{-modules}\}\\ (F_n)_{n\in\mathbb{N}} &\longmapsto \lim_n F_n\end{aligned}$$

is an equivalence of categories with quasi-inverse

$$\psi(M) := (M/I^nM)_{n\in\mathbb{N}} \longleftarrow M$$

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be an AR I-adic system of R-modules. Without loss of generality assume $(F_n)_{n\in\mathbb{N}}$ is an I-adic system. Let $F := \lim_n F_n$, which is a finite R-module by Proposition 1.3.7. The same proposition shows that $F/I^{n+1}F \cong F_n$. Hence, $\psi \circ \phi \cong \mathrm{id}$. Conversely, assume M is a finite R-module. Then $\phi \circ \psi(M)$ is by definition the I-adic completion $M^\wedge$ of M. Since R is noetherian and I-adically complete, Proposition 1.3.8 implies

$$M^\wedge \cong M \otimes_R R^\wedge \cong M \otimes_R R \cong M$$

which completes the proof. □

The later results heavily rely on the following well-known and essential lemma.

Lemma 3.2.5 (Artin-Rees Lemma)**.** Let R be a noetherian ring and M a finitely generated R-module. Further let $N \subset M$ be a submodule. Then there exists an integer $r \in \mathbb{N}$ such that for all $n \geq r$ the equality

$$I^n M \cap N = I^{n-r}(I^r M \cap N)$$

holds.

Proof. [4, Lemma 5.1] □

Proposition 3.2.6. Assume R is noetherian. Let

$$f : (M_n)_{n\in\mathbb{N}} \to (N_n)_{n\in\mathbb{N}}$$

be a morphism of AR I-adic modules. Then the kernel and the cokernel of this map in the AR-category are again AR I-adic. In particular, the subcategory

$$\{\text{AR } I\text{-adic systems of } R\text{-modules}\} \subset \mathrm{AR}(\mathrm{Mod}_R)$$

is an abelian subcategory.

Proof. ([6, Proposition 10.1.4 (iii)]) As the completion $R^\wedge$ is noetherian and the formula $R^\wedge/I^n R^\wedge \cong R/I^n R$ holds for any $n \in \mathbb{N}$, we can assume that R is I-adically complete (replace R by $R^\wedge$). Moreover, we can assume that $(M_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ are I-adic systems.
By Lemma 3.2.3 it follows that f is represented by an element

$$(f_n)_{n\in\mathbb{N}} \in \mathrm{Hom}((M_n)_{n\in\mathbb{N}}, (N_n)_{n\in\mathbb{N}}).$$

This means that f is given by a family of morphisms $f_n : M_n \to N_n$, which are compatible with the transition maps $M_{n+1} \to M_n$, respectively $N_{n+1} \to N_n$. Moreover, the kernel and cokernel of f in the AR-category are given by $(\ker(f_n))_{n\in\mathbb{N}}$ and $(\mathrm{coker}(f_n))_{n\in\mathbb{N}}$.
We apply Proposition 3.2.4 to get finitely generated R-modules M, N and a map $g : M \to N$ such that

$$M/I^{n+1}M \cong M_n \quad \text{and} \quad N/I^{n+1}N \cong N_n.$$

Further $f_n : M/I^{n+1}M \to N/I^{n+1}N$ is induced by g. Finally, this preliminary work gives access to the main part of the proof.
We will begin with the easier part and demonstrate that the cokernel of $(f_n)_{n\in\mathbb{N}}$ is I-adic[1]. We have an exact sequence

$$M \xrightarrow{g} N \to \mathrm{coker}(g) \to 0,$$

[1] It is not only isomorphic to an I-adic system.

which induces for any $n \geq 0$ an exact sequences

$$M/I^{n+1}M \xrightarrow{g} N/I^{n+1}N \longrightarrow \mathrm{coker}(g)/I^{n+1}\mathrm{coker}(g) \longrightarrow 0.$$

Consequently, $\mathrm{coker}(f_n) = \mathrm{coker}(g)/I^{n+1}\mathrm{coker}(g)$. Since $\mathrm{coker}(g)$ is a finitely generated module, this means that $(\mathrm{coker}(f_n))_{n\in\mathbb{N}}$ is an I-adic system of R-modules.
We continue by showing that the system of kernels $(\ker(f_n))_n$ is AR-isomorphic to an I-adic system. Consider the morphism

$$\ker(f)/I^{n+1}\ker(f) \to M/I^{n+1}M \xrightarrow{f_n} N/I^{n+1}N.$$

One easily sees that this composite is zero and hence the universal property of the kernel induces a map $\ker(f)/I^{n+1}\ker(f) \xrightarrow{\psi_n} \ker(f_n)$. We have to show that $(\ker(\psi_n))_n$ and $(\mathrm{coker}(\psi_n))_n$ form null systems. To establish this for $(\mathrm{coker}(\psi_n))_n$, we must find an integer r such that for all $n \in \mathbb{N}$ the image of an element $x + I^{n+1+r}M \in \ker(f_{n+r}) \subset M/I^{r+n+1}M$ in $M/I^{n+1}M$ comes from an element in $\ker(f)/I^{n+1}\ker(f)$. Choose r such that the Artin-Rees lemma is fulfilled for N and the submodule $\mathrm{im}(f) \subset N$. That is, choose r with

$$I^nN \cap \mathrm{im}(f) = I^{n-r}(I^rN \cap \mathrm{im}(f))$$

for all $n \geq r$. Let $x + I^{n+1+r}M \in \ker(f_{n+r})$ for a fixed $x \in M$. We deduce that

$$f(x) \in I^{n+1+r}N \cap \mathrm{im}(f) = I^{n+1}(I^rN \cap \mathrm{im}(f)).$$

This gives immediately an element $x' \in M$ and a $\lambda \in I^{n+1}$ such that $f(x) = \lambda f(x')$. Hence $x - \lambda x'$ is an element of $\ker(f)$ and represents the same equivalence class as x modulo $I^{n+1}M$. Hence $x + I^{n+1}M \in M/I^{n+1}M$ comes from an element in $\ker(f)/I^{n+1}\ker(f)$.
Similarly, we will show that $(\ker(\psi_n))_n$ is a null system. With the Artin-Rees lemma we can find an $r \in \mathbb{N}$ with

$$\ker(f) \cap I^nM = I^{n-r}(\ker(f) \cap I^rM)$$

for all $n \geq r$. Let $x + I^{n+1+r}\ker(f) \in \ker(\psi_{n+r}) \subset \ker(f)/I^{n+1+r}\ker(f)$ for a fixed $x \in \ker(f)$. Hence, $[x]$ is zero in $M/I^{n+1+r}M$ and therefore

$$x \in \ker(f) \cap I^{n+1+r}M = I^{n+1}(\ker(f) \cap I^rM).$$

This gives an $x' \in \ker(f)$ and a $\lambda \in I^{n+1}$ with $x = \lambda x'$. Hence the image of x in $\ker(f)/I^{n+1}\ker(f)$ is zero. But this exactly means that the system $(\ker\psi_n)_{n\in\mathbb{N}}$ is a null system. We showed that $(\ker\psi_n)_{n\in\mathbb{N}}$ and $(\mathrm{coker}\psi_n)_{n\in\mathbb{N}}$ are null sytems and hence, ψ is an isomorphism. $\square$

3.3 Adic Systems of Sheaves

We can introduce the formalism of Section 3.2 for sheaves of R-modules. However, in this context it is not possible to pass to the limit like in Proposition 3.2.4. Again fix for this subsection a noetherian ring R, where $I \subset R$ is an ideal. We introduce analogously to Definition 3.2.1:

Definition 3.3.1. An inverse system $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of sheaves of R-modules is called *I-adic system* (or by abuse of notation *I-adic sheaf*) if each $\mathcal{F}_n$ is constructible and the following properties are fulfilled.

- $I^{n+1}\mathcal{F}_n = 0$
- The transition maps $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induce isomorphisms $\mathcal{F}_{n+1}/I^{n+1}\mathcal{F}_{n+1} \cong \mathcal{F}_n$.

Now consider the AR-category $\mathrm{AR}(\mathrm{Mod}_R(X_{\mathrm{et}}))$. An object in this category is called *AR I-adic sheaf* if it is isomorphic in the AR-category to an I-adic system.

Remark 3.3.2. Although it seems confusing, the literature often uses the term I-adic sheaf for I-adic systems. As this is standard, we adopt this notation. The reader shoud be alerted that an I-adic sheaf on the étale site is an inverse system of sheaves and not an sheaf in the usual sense. In the pro-étale world the formalism improves and the I-adic sheaves are actual sheaves of R-modules.

Examples 3.3.3. Let X be a scheme.

- Let M be a finitely generated R-module and $\mathcal{F}_n := \underline{M/I^{n+1}M}$ the constant sheaf associated to $M/I^{n+1}M$ on X_{et}. Then the system $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is an I-adic sheaf.
- Let l be a prime number and $X := \mathbb{Z}_l$. Then $(\mu_{X,l^n}, u_n)_{n\in\mathbb{N}}$ is an l-adic sheaf, where the transition maps u_n are defined by $s \mapsto s^l$.
- If $\mathcal{F}$ is a constructible sheaf on X_{et} then $(\mathcal{F}/I^{n+1}\mathcal{F})_{n\in\mathbb{N}}$ is an I-adic sheaf of R-modules. It is not true that any I-adic sheaf is of this form.

Lemma 3.3.4. Let $\mathcal{F}$ be an I-adic sheaf and $\mathcal{G}$ an inverse system in $\mathrm{Mod}_R(X_{\mathrm{et}})$ with $I^{n+1}\mathcal{G} = 0$, then

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}_{\mathrm{AR}}(\mathcal{F},\mathcal{G}).$$

In particular, the category of I-adic sheaves is in equivalence with the full subcategory of AR I-adic sheaves.

Proof. This can be proven analogously to Lemma 3.2.3. □

Definition 3.3.5. An inverse system of étale sheaves $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ is called *locally constant* or *lisse* if all $\mathcal{F}_n$ are locally constant. Further define the restriction $\mathcal{F}|_{X'}$ (for suitable X') as the system $(\mathcal{F}_n|_{X'})_{n\in\mathbb{N}}$.

Lemma 3.3.6. Let Λ be a noetherian ring and $I \subset \Lambda$ an ideal. Then we get a noetherian ring $R := \bigoplus_{i\in\mathbb{N}} I^n/I^{n+1}$ (by definition we assume $I^0 = \Lambda$) whose addition is the obvious one. The multiplication is given by the rule

$$\begin{aligned} I^n \times I^m &\longmapsto I^{m+n} \\ (x,y) &\longmapsto xy, \end{aligned}$$

which uniquely extends to a multiplication on R.

Proof. It is easy to check that R is a ring. To check that R is noetherian choose generators $f_1,\dots,f_n$ of the ideal I. It is clear that $f_1,\dots,f_n \in I/I^2 \subset R$ generate R as Λ-algebra. Hence, we obtain a surjective algebra homomorphism

$$\Lambda[X_1,\dots,X_n] \to R,$$

which proves that R is noetherian. □

The only proposition for which the proof from [6] or [5] can not be adjusted is the following.

Proposition 3.3.7. Let X be a noetherian scheme, Λ a noetherian ring and $I \subset \Lambda$ an ideal. Further, let $F = (F_n)_{n\in\mathbb{N}}$ be an I-adic sheaf on X_{et}. Then there exists a finite partition $X = \bigsqcup_{i=1}^n X_i$ such that $F|_{X_i}$ is locally constant.

Proof. We will use the idea of [15, Tag 09BU] adapted to the étale version. Consider the noetherian ring $R := \bigoplus_{i\in\mathbb{N}} I^n/I^{n+1}$ from Lemma 3.3.6. By assumption F_n is a constructible sheaf of Λ/I^{n+1}-modules for all $n \in \mathbb{N}$. As constructibility is preserved under taking images and tensors, the sheaf

$$I^n F_n = \operatorname{im}(\underline{I^n} \otimes_\Lambda F_n \to F_n)$$

is a constructible sheaf of Λ/I^{n+1}-modules. It is easy to see that $I^n F_n$ is also a sheaf of $\Lambda/I\Lambda$-modules and, of course, constructible as a sheaf of $\Lambda/I\Lambda$-modules[2]. Define

$$G := \bigoplus_{i\in\mathbb{N}} I^n F_n$$

and equip G with the canonical R-module structure. As X is noetherian, we can make use of Corollary 2.3.8 and see that the constructible sheaves form a Serre subcategory of all sheaves of R-modules on X_{et}. In particular, the surjection

$$\underline{R} \otimes_{\Lambda/I} F_0 \twoheadrightarrow G$$

implies that G is a constructible sheaf of R-modules. Choose a finite partition $X = \bigsqcup_i X_i$ such that $G|_{X_i}$ is locally constant. Without loss of generality we assume $X = X_i$ and we

[2] By definition $I^0 := R$.

want to show that F_n is locally constant for all n. By assumption G is locally constant with finitely generated stalks[3], so by Corollary 3.3.8 all specialization morphisms

$$G_{x'} \to G_x$$

are isomorphisms. As taking stalks commutes with the direct sum, it follows immediately that all specialization morphisms $(I^n F_n)_{x'} \to (I^n F_n)_x$ have to be isomorphisms. Proposition 2.2.3 implies that $I^n F_n$ is locally constant for all n. Especially, F_0 is locally constant and we can proceed by induction. For the induction step, consider the short exact sequnce

$$0 \to I^n F_n \to F_n \to F_{n-1} \to 0$$

and apply Lemma 1.1.29 to show that F_n is locally constant, which completes the proof. □

Corollary 3.3.8. Let X be a noetherian scheme, Λ a noetherian ring and $I \subset \Lambda$ an ideal. Let $\mathcal{F} = (F)_{n\in\mathbb{N}}$ an I-adic sheaf. Then there exists an open dense subset $U \subset X$ such that $\mathcal{F}|_U$ is lisse.

Proof. Since X is noetherian, there are finitely many irreducible components $X_1, \dots, X_n$ of X. Define $U_i := X \setminus \bigcup_{j\neq i} X_j$, which is an open irreducible subset of X. Further U_i contains the generic point of X_i which implies that the union $\bigcup_i U_i$ is a dense open subset of X. Hence, we can assume that X is irreducible with generic point η. By Corollary 4.4.15 there is a finite partition $X = \bigsqcup_i X_i$ such that $\mathcal{F}|_{X_i}$ is lisse. Choose i with $\eta \in X_i$ and find a factorization of the inclusion $X_i \subset X$

$$X_i \overset{\iota}{\hookrightarrow} U \overset{j}{\hookrightarrow} X$$

with ι a closed and j an open immersion. Since X_i contains the generic point η of X, which is also a generic point of U, we conclude that $\iota = \mathrm{id}$. In particular, X_i is an open dense subset of X such that $\mathcal{F}|_{X_i}$ is locally constant. □

Definition 3.3.9. Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ an inverse system of sheaves of R-modules. For a geometric point $s : \mathrm{Spec}(\Omega) \to X$ define the *stalk at s* as

$$\mathcal{F}_s := \lim_n \mathcal{F}_{n,s}$$

and the *system of stalks at s* as the inverse system of R-modules given by $(\mathcal{F}_{n,s})_{n\in\mathbb{N}}$.

The assertion $\mathcal{F} \mapsto (\mathcal{F}_{n,s})_{n\in\mathbb{N}}$ is exact and maps null systems to null systems. By the universal property of the AR-category (Proposition 1.2.3) it induces an exact functor from $\mathrm{AR}(\mathrm{Mod}_R(X_{\mathrm{et}}))$ into the category $\mathrm{AR}(\mathrm{Mod}_R)$.

Lemma 3.3.10. If $\mathcal{F}$ is an AR I-adic sheaf then the system of stalks $(\mathcal{F}_{n,s})_{n\in\mathbb{N}}$ is AR I-adic.

[3] As R-modules.

Proof. There is an AR-isomorphism $\mathcal{F} \cong \mathcal{F}'$ for an I-adic sheaf $\mathcal{F}'$. We assume $\mathcal{F}$ is I-adic and show that $(\mathcal{F}_{n,s})_{n\in\mathbb{N}}$ is I-adic.
Obviously, $\mathcal{F}_{n,s}$ is a finitely generated R-module for any $n \in \mathbb{N}$. Moreover, there are isomorphisms

$$\mathcal{F}_{n+1}/I^n\mathcal{F}_{n+1} \xrightarrow{\cong} \mathcal{F}_n.$$

For any geometric point s this translates into an isomorphism

$$\mathcal{F}_{n+1,s}/I^n\mathcal{F}_{n+1,s} \xrightarrow{\cong} \mathcal{F}_{n,s},$$

which proves the claim. □

For the rest of this chapter assume, if not otherwise stated, that X is noetherian. We are now able to introduce an easier criterion to detect an AR I-adic sheaf, although it looks more complicated at a first look.

Proposition 3.3.11. Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ be an inverse system of constructible sheaves on a noetherian scheme X. Further assume $I^{n+1}\mathcal{F}_n = 0$ for all $n \in \mathbb{N}$. Then being an AR I-adic sheaf is equivalent to fulfilling ARML (Definition 3.1.3) and the following condition, which we call ARML2: Let $r \in \mathbb{N}$ such that for all $t \geq r$ we have

$$\operatorname{im}(\mathcal{F}[t] \to \mathcal{F}) = \operatorname{im}(\mathcal{F}[r] \to \mathcal{F}),$$

which exists because of the condition ARML. Then we demand the existence of an integer $s \in \mathbb{N}$ such that for $t \geq s$ the transition maps $\mathcal{F}_{n+t} \to \mathcal{F}_{n+s}$ induce isomorphisms

$$\overline{\mathcal{F}_{n+t}}/I^{n+1}\overline{\mathcal{F}_{n+t}} \cong \overline{\mathcal{F}_{n+s}}/I^{n+1}\overline{\mathcal{F}_{n+s}},$$

for all $n \in \mathbb{N}$. Here $\overline{\mathcal{F}_n} := \operatorname{im}(\mathcal{F}_{n+r} \to \mathcal{F}_n)$.
Moreover, $(\overline{\mathcal{F}_{n+s}})_{n\in\mathbb{N}}$ is then an I-adic sheaf which is AR-isomorphic to $\mathcal{F}$.

Proof. We will concentrate on the implication that ARML and ARML2 imply that $\mathcal{F}$ is AR I-adic. The other direction can be found for $R = \mathbb{Z}_l$ and $I = (l)$ in [6, Prop. 10.1.1]. The minor adaptions for the proof are left to the reader.
Assume ARML and ARML2 are fulfilled and choose r and s as in the assumptions. Define $\overline{\mathcal{F}} := (\overline{\mathcal{F}_n})_{n\in\mathbb{N}}$ and note that this system is AR-isomorphic to $\mathcal{F}$ via the canonical map $\overline{\mathcal{F}} \hookrightarrow \mathcal{F}$. Its inverse is given by $\mathcal{F}[r] \to \mathcal{F}$ (see also Lemma 3.1.7). Further we can give an AR-isomorphism

$$\overline{\mathcal{F}} \cong_{\mathrm{AR}} (\overline{\mathcal{F}_{n+s}}/I^{n+1}\overline{\mathcal{F}_{n+s}})_{n\in\mathbb{N}}.$$

Precisely, as $I^{n+1}\mathcal{F}_n = 0$ by assumption and $\overline{\mathcal{F}_n} \subset \mathcal{F}_n$, there is a canonical morphism $(\overline{\mathcal{F}_{n+s}}/I^{n+1}\overline{\mathcal{F}_{n+s}})_{n\in\mathbb{N}} \to \overline{\mathcal{F}}$. Its inverse in the AR category is given by the canonical projection $\overline{\mathcal{F}}[s] \to (\overline{\mathcal{F}_{n+s}}/I^{n+1}\overline{\mathcal{F}_{n+s}})_{n\in\mathbb{N}}$. We achieved that

$$\mathcal{F} \cong_{\mathrm{AR}} (\overline{\mathcal{F}_{n+s}}/I^{n+1}\overline{\mathcal{F}_{n+s}})_{n\in\mathbb{N}}$$

and it remains to show that the right side defines an I-adic sheaf. As the sheaves $\mathcal{F}_n$ are constructible for all $n \in \mathbb{N}$, so are $\mathcal{G}_n := \overline{\mathcal{F}_{n+s}}/I^{n+1}\overline{\mathcal{F}_{n+s}}$. Using the choice of s, we can further compute

$$\begin{aligned}\mathcal{G}_{n+1}/I^{n+1}\mathcal{G}_{n+1} &\cong \overline{\mathcal{F}_{n+1+s}}/I^{n+1}\overline{\mathcal{F}_{n+1+s}}\\ &\cong \overline{\mathcal{F}_{n+s}}/I^{n+1}\overline{\mathcal{F}_{n+s}}\\ &= \mathcal{G}_n.\end{aligned}$$

□

Remark 3.3.12. One can easily adapt Proposition 3.3.11 to work for I-adic systems of R-modules. In the assumptions, $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ has then to be a system of finitely generated R-modules with $I^{n+1}\mathcal{F}_n = 0$. The statement stays the same replacing the term "sheaf" by "R-module".

The main advantage of Proposition 3.3.11 is that we can check all the conditions "locally". This gives the possibility to check if a sheaf is AR I-adic via the use of noetherian induction or to consider in some cases the system of stalks defined in Definition 3.3.9. We make this precise in the following lemma.

Lemma 3.3.13. Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ be an inverse system of constructible schemes on X and further assume $I^{n+1}\mathcal{F}_n = 0$ for all $n \in \mathbb{N}$. Then $\mathcal{F}$ is AR I-adic if and only if there exists integers r and s, such that for any stalk ξ one can show that $(\mathcal{F}_{n,\xi})_{n\in\mathbb{N}}$ fulfills ARML and ARLM2 using the fixed integers r and s for Proposition 3.3.11.

Proof. Taking stalks is exact and commutes with the tensor product. Therefore, for a geometric point ξ we have the equalities

$$\left(\overline{\mathcal{F}_n}\right)_\xi = \overline{\mathcal{F}_{n\xi}} \quad \text{and} \quad \left(I^n\overline{\mathcal{F}_n}\right)_\xi = I^n\overline{\mathcal{F}_{n\xi}}.$$

Further, checking that a morphism is an isomorphism can be done on stalks. It is now a straightforward calculation to show the desired result. □

Lemma 3.3.14. Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ be a system of locally constant sheaves of R-modules with finitely generated stalks. Further assume that X is noetherian and connected. Then for any two geometric points s and s' of X there is an isomorphism of systems of R-modules

$$(\mathcal{F}_{n,s})_{n\in\mathbb{N}} \cong (\mathcal{F}_{n,s'})_{n\in\mathbb{N}}.$$

Moreover, for any $n \in \mathbb{N}$ the isomorphisms $\mathcal{F}_{n,s} \cong \mathcal{F}_{n,s'}$ come from the composition of specialization maps and their inverses.

Proof. Let s, s' be two geometric points and fix a natural number n. If the images x and x' of s and s' lie in the same irreducible component we have $\mathcal{F}_{n,s} \cong \mathcal{F}_{n,s'}$ by Lemma 2.2.1 and Proposition 2.2.3. Concretely, let η the generic point of this irreducible component[4]. Then by Lemma 2.2.1 and Proposition 2.2.3 we have specialization maps

$$\mathcal{F}_{n,s} \xrightarrow{\cong} \mathcal{F}_{n,\eta} \quad \text{and} \quad \mathcal{F}_{n,s'} \xrightarrow{\cong} \mathcal{F}_{n,\eta}.$$

Since X is connected by assumption, two irreducible components $Y, Y' \subset X$ are connected via a finite chain $Y = Y_1, \ldots, Y_n = Y'$ of irreducible subsets[5], such that the intersection $Y_i \cap Y_{i+1}$ is non-empty. This proves $\mathcal{F}_{n,s} \cong \mathcal{F}_{n,s'}$, even if s and s' are in different irreducible components. Moreover, this isomorphism is the composition of specialization maps and their inverses. To conclude, we have to show that these isomorphisms $\mathcal{F}_{n,s} \cong \mathcal{F}_{n,s'}$ give rise to an isomorphism of systems of R-modules $(\mathcal{F}_{n,s})_{n\in\mathbb{N}} \cong (\mathcal{F}_{n,s'})_{n\in\mathbb{N}}$. Therefore it remains to prove the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F}_{n,s} \arrow[r] \arrow[d] & \mathcal{F}_{n,\eta} \arrow[d] \\
\mathcal{F}_{n-1,s} \arrow[r] & \mathcal{F}_{n-1,\eta},
\end{tikzcd}$$

where the horizontal arrows are specialization maps. This follows directly from the construction in Lemma 2.2.2. □

The easy but essential corollary from [5, p. 124] is the following:

Corollary 3.3.15. Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ be an inverse system of constructible sheaves with $I^{n+1}\mathcal{F}_n = 0$. Then the following statements hold

1. Assume $\mathcal{F}$ is AR I-adic. For any morphism of noetherian schemes $f : X' \to X$ the system $f^*\mathcal{F} = (f^*\mathcal{F}_n)_{n\in\mathbb{N}}$ is an AR I-adic sheaf.

2. Let $(U_i \to X)_{i\in\{1,\ldots,n\}}$ be a finite étale cover of X. Then $\mathcal{F}$ is an AR I-adic sheaf if and only if $\mathcal{F}|_{U_i}$ is an AR I-adic sheaf on U_i for all i.

3. Let $A \subset X$ be a closed subscheme and $U := X \setminus A$. Then $\mathcal{F}$ is AR I-adic if and only if

$$\mathcal{F}|_U \quad \text{and} \quad \mathcal{F}|_A$$

   are AR I-adic.

4. Assume $\mathcal{F}_n$ is locally constant for all $n \in \mathbb{N}$ and X is a connected and noetherian scheme. Then $\mathcal{F}$ is an AR I-adic sheaf if and only if $(F_{n,s})_{n\in\mathbb{N}}$ is an AR I-adic system of R-modules for at least one geometric point $s : \mathrm{Spec}(\Omega) \to X$.

[4] I.e. $x, x' \in \overline{\{\eta\}}$

[5] The Y_i can be chosen to be the irreducible components of X.

Proof. We start with the first claim. Note that the constructibility of $f^*\mathcal{F}_n$ is covered by Proposition 2.3.4. Since $\mathcal{F}$ is AR I-adic we can find integers r and s such that the two conditions ARML and ARML2 in Proposition 3.3.11 are fulfilled. We claim that these integers also work for $f^*\mathcal{F}$. Checking that r and s are suitable integers for Proposition 3.3.11 can be done on stalks by Lemma 3.3.13. For any geometric point ξ of X' we use the equality $f^*\mathcal{F}_\xi \cong \mathcal{F}_{f(\xi)}$ in Proposition 2.1.9 to conclude.
For 2 first assume that $\mathcal{F}$ is AR I-adic. It follows directly from 1 that $\mathcal{F}|_{U_i}$ is AR I-adic for all i. Conversely, assume $\mathcal{F}|_{U_i}$ is I-adic and consider the integers r_i and s_i as in the notion of Proposition 3.3.11. Define

$$r = \max_i \{r_i\} \quad \text{and} \quad s = \max_i \{s_i\}.$$

A stalkwise calculation easily verifies that ARML and ARML2 are fulfilled using r and s.
For 3 it is trivial to show that $\mathcal{F}|_U$ and $\mathcal{F}|_A$ are AR I-adic, if $\mathcal{F}$ is so. The other direction works analogously to 2, using that any geometric point factors either through A or through U.
To show 4, first assume $(\mathcal{F}_{n,\xi})_{n\in\mathbb{N}}$ is an AR I-adic system of R-modules at one geometric point ξ. Choose suitable integers r and s such that ARML and ARML2 from Proposition 3.3.11 are fulfilled for this particular system. For any other geometric point η Lemma 3.3.14 implies

$$(\mathcal{F}_{n,\xi})_{n\in\mathbb{N}} \cong (\mathcal{F}_{n,\eta})_{n\in\mathbb{N}}$$

and, in particular, that r and s work as integers for ARML and ARML2 on any stalk. Lemma 3.3.13 finally proves the claim. Conversely, if $\mathcal{F}$ is an AR I-adic sheaf then $(\mathcal{F}_{n,s})_{n\in\mathbb{N}}$ is an AR I-adic system of R-modules by Lemma 3.3.10. □

3.4 The AR-category of Adic Systems

In this section fix a noetherian scheme X and a noetherian ring R. We investigate in the subcategory of $\mathrm{AR}(\mathrm{Mod}_R(X_{\mathrm{et}}))$ consisting of AR I-adic sheaves.

Proposition 3.4.1. Let $f : \mathcal{F} \to \mathcal{G}$ be an AR-morphism between two AR I-adic sheaves. Then $\ker(f)$ and $\operatorname{coker}(f)$ are AR I-adic. In particular, the category of AR I-adic sheaves is an abelian subcategory of the AR-category of sheaves of R-modules.

Proof. Without loss of generality we can assume that $\mathcal{F}$ and $\mathcal{G}$ are I-adic sheaves. By Corollary 3.3.15 3 we can assume that X is irreducible and go on by noetherian induction. To be precise, let

$$M := \{A \mid A \text{ is a closed subscheme of } X \text{ such that } \ker(f)|_A \text{ is not AR } I \text{ adic}\}$$

and assume for a contradiction that $X \in M$. If we can find an open non-empty subset $U \subset X$ such that $\ker(f)|_U$ is AR I-adic, then we deduce from Corollary 3.3.15

3 that $A := X \setminus U \in M$. But since restriction is an exact functor we obtain that $\ker(f)|_A = \ker(f|_A : \mathcal{F}|_A \to \mathcal{G}|_A)$ is again the kernel of a morphism between AR I-adic sheaves. Replacing X by A we can find an infinitely long decreasing chain of closed subsets of X, which contradicts the noetherian assumption. One proceeds similarly for $\operatorname{coker}(f)$.
Hence, we have to find an open non-empty subset $U \subset X$ such that $\ker(f)|_U$ and $\operatorname{coker}(f)|_U$ are AR I-adic. By Proposition 3.3.7 we can choose an open dense[6] subset U in X such that $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are lisse. Hence, $\ker(f)|_U$ and $\operatorname{coker}(f)|_U$ are lisse by Lemma 1.1.29. Then we can use Corollary 3.3.15 4 and equivalently show that the system of stalks at one single stalk is AR I-adic. This proves the claim since taking stalks is exact and kernels and cokernels of AR I-adic systems of R-modules are AR I-adic by Proposition 3.2.6. □

Lemma 3.4.2. Assume R is complete. Any geometric point x gives rise to an exact functor

$$\begin{aligned}(-)_x : \{\text{AR } I\text{-adic sheaves}\} &\to \{\text{finitely generated } R\text{-modules}\}\\ \mathcal{G} = (G_n)_{n\in\mathbb{N}} &\to \mathcal{G}_x := \lim_n G_{n,x}.\end{aligned}$$

Moreover, an AR I-adic sheaf $\mathcal{F}$ is AR-zero if and only if $\mathcal{F}_x = 0$ for any geometric point x.

Proof. By Proposition 3.2.4, the limit functor induces an equivalence of categories between AR I-adic systems of R-modules and the category of finitely generated R-modules. The functor $(-)_x$ is the composition of the exact functor

$$\begin{aligned}\{\text{AR } I\text{-adic sheaves}\} &\to \{\text{AR } I\text{-adic systems of modules}\}\\ (G_n)_{n\in\mathbb{N}} &\to (G_{n,x})_{n\in\mathbb{N}}\end{aligned}$$

with the mentioned equivalence of categories. This implies the exactness of $(-)_x$.
Assume, $\mathcal{F}$ is an AR I-adic sheaf such that $\mathcal{F}_x = 0$ for all geometric points x. Without loss of generality we can assume that $\mathcal{F}$ is I-adic. Now, Proposition 1.3.7 implies that the system of stalks $(\mathcal{F}_{n,x})_{n\in\mathbb{N}}$ is zero[7]. But this implies $\mathcal{F}_n = 0$ for all $n \in \mathbb{N}$ and proves the claim. □

Proposition 3.4.3. The category of AR I-adic sheaves is noetherian. That is, if $\mathcal{F}$ is an AR I-adic sheaf and

$$\mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \ldots \subset \mathcal{F}^{(n)} \subset \ldots \subset \mathcal{F}$$

is an increasing chain of AR I-adic subsheaves of $\mathcal{F}$, then this chain gets eventually constant. To clearify notation, an AR I-adic subsheaf $\mathcal{H}$ of $\mathcal{F}$ is an AR-injective map $\mathcal{H} \hookrightarrow \mathcal{F}$ and eventually constant means that the injections $\mathcal{F}^{(n)} \hookrightarrow \mathcal{F}^{(n+1)}$ get AR-isomorphisms for large n.

[6] In particular non-empty.
[7] Not only AR-zero.

Proof. Again we assume that X is irreducible and go on by noetherian induction. Concretely, let

$$M := \left\{ A \;\middle|\; \begin{array}{l} A \text{ is a closed subscheme of } X \text{ such that} \\ \mathcal{F}^{(0)}|_A \subset \ldots \subset \mathcal{F}^{(n)}|_A \ldots \text{ does not get eventually constant} \end{array} \right\}$$

and assume for a contradiction that $X \in M$. If we can find an open non-empty subset $U \subset X$ such that $\mathcal{F}^{(0)}|_U \subset \ldots \subset \mathcal{F}^{(n)}|_U$ gets eventually constant, then we definitly have $A := X \setminus U \in M$. Continuing with A instead of X yields to an infinitely long decreasing chain of closed subsets in M, which contradicts the noetherian assumption on X.
Therefore, we have to find an open subset $U \subset X$ such that

$$\mathcal{F}^{(0)}|_U \subset \mathcal{F}^{(1)}|_U \subset \ldots$$

gets eventually constant.
Let η be a generic point of X and apply Lemma 3.4.2 to get an increasing chain of finitely generated $R^\wedge$-modules

$$\mathcal{F}^{(1)}_\eta \subset \ldots \subset \mathcal{F}^{(n)}_\eta \subset \ldots \subset \mathcal{F}_\eta.$$

The ring $R^\wedge$ is noetherian, as R is noetherian by assumption. We conclude that the above chain gets eventually constant. Fix an $m \gg 0$, such that $\mathcal{F}^{(m)}_\eta = \mathcal{F}^{(m+r)}_\eta$ for all $r \in \mathbb{N}$. Proposition 3.4.1 shows that $\mathcal{F}/\mathcal{F}^{(m)}$ is an AR I-adic sheaf. By choosing an isomorphism to an I-adic sheaf we can assume that $\mathcal{F}/\mathcal{F}^{(m)}$ is I-adic itself. Applying Proposition 3.3.7 yields an open dense subset $U \subset X$ such that $\mathcal{F}/\mathcal{F}^{(m)}|_U$ is locally constant.
We claim that $\left(\mathcal{F}^{(n)}/\mathcal{F}^{(m)}\right)|_U$ is AR-zero, which we wanted to show. By Lemma 3.4.2 it is enough to check

$$\mathcal{F}^{(n)}_x/\mathcal{F}^{(m)}_x = 0$$

for all geometric points x over U. Using Lemma 2.2.1 and Proposition 2.2.3 one notices that for any geometric point x over U the specialization maps induce isomorphisms

$$\mathcal{F}_x/\mathcal{F}^{(m)}_x \xrightarrow{\cong} \mathcal{F}_\eta/\mathcal{F}^{(m)}_\eta.$$

Consider the diagram

$$\begin{tikzcd}
\mathcal{F}^{(n)}_x/\mathcal{F}^{(m)}_x \arrow[r, "\varphi"] \arrow[d, hook] & \mathcal{F}^{(n)}_\eta/\mathcal{F}^{(m)}_\eta \arrow[d, hook] \\
\mathcal{F}_x/\mathcal{F}^{(m)}_x \arrow[r, "\cong"] & \mathcal{F}_\eta/\mathcal{F}^{(m)}_\eta,
\end{tikzcd}$$

where the horizontal arrows are the specialization maps and the vertical maps come from the inclusions. The diagram can be easily verified as commutative using the construction in Lemma 2.2.2. By the choice of m, we have $0 = \mathcal{F}^{(n)}_\eta/\mathcal{F}^{(m)}_\eta$. As φ is injective, $\mathcal{F}^{(n)}_x/\mathcal{F}^{(m)}_x = 0$ must hold. This proves the claim. $\square$

4 Sheaves on the Pro-étale Site

This is the main chapter of this thesis. In Section 4.1 and Section 4.2 some preliminaries about topoi and homological algebra are covered. These are followed by the axioms of a pro-étale enlargement in Section 4.3 and the statements about adic pro-étale sheaves in Section 4.4. In Section 4.6, we draw a connection to constructible complexes defined by B. Bhatt and P. Scholze in [2]. Finally, the last section gives an outlook of two possible constructions of a pro-étale enlargement.

4.1 A Digression in Homological Algebra

In the following let $\mathcal{A}$ be an abelian category. This section can be seen as second part of Section 3.1. We will investigate in injective objects in the categories $\mathcal{A}^{\mathbb{N}}$ and $\mathrm{AR}(\mathcal{A})$. We will later use the results to transfer the statements of Chapter 3 to the theory of adic pro-étale sheaves.

Lemma 4.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \to \mathcal{B}$ a functor which admits an exact left adjoint $L : \mathcal{B} \to \mathcal{A}$. Under this conditions, F preserves injective objects.

Proof. Let $\mathcal{I} \in \mathcal{A}$ be an injective object. Then the adjunction (L, F) gives rise to an isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(-, F(\mathcal{I})) \cong \mathrm{Hom}_{\mathcal{A}}(L(-), \mathcal{I}) = \mathrm{Hom}_{\mathcal{A}}(-, \mathcal{I}) \circ L$$

which is an exact functor as it is the composition of two exact functors. By definition $F(\mathcal{I})$ is an injective object. □

Proposition 4.1.2. Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}^{\mathbb{N}}$ has enough injectives if and only if $\mathcal{A}$ has enough injectives. Moreover, a system $(A_n, d_n) \in \mathcal{A}^{\mathbb{N}}$ is injective if and only if all A_n are injective and the transition maps are split surjections.

Proof. ([10, Proposition 1.1]) Assume $\mathcal{A}^{\mathbb{N}}$ has enough injectives. For $m \in \mathbb{N}$ there is a pair of adjunction

$$U_m : \mathcal{A}^{\mathbb{N}} \leftrightarrows \mathcal{A} : V_m$$

given by $U_m((A_n, d_n)_n) := A_m$ and

$$V_m(A) := 0 \to \cdots 0 \to \underbrace{A \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} A}_{m-\text{times}}$$

Observe, that U_m is exact and that $V_m U_m(A) = A$ for any $A \in \mathcal{A}$. For $A \in \mathcal{A}$ choose a monomorphism $U_m(A) \hookrightarrow \mathcal{I}$ in an injective object of $\mathcal{A}^{\mathbb{N}}$. By Lemma 4.1.1 we obtain that $A = V_m U_m(A) \hookrightarrow V_m \mathcal{I}$ is a monomorphism in the injective object $V_m \mathcal{I} \in \mathcal{A}$. This implies further, that for an injective object $(I_n)_{n\in\mathbb{N}}$ in $\mathcal{A}^{\mathbb{N}}$ all I_n are injective.
Conversly, assume $\mathcal{A}$ has enough injectives. We will proceed in a similar fashion as before. Consider the product category $\mathcal{A}^{|\mathbb{N}|}$ whose objects are families of the form $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathcal{A}$ and whose morphisms are defined componentwise. It is trivial to show that $\mathcal{A}^{|\mathbb{N}|}$ has enough injectives[1]. Again we define a pair of adjunction as

$$V : \mathcal{A}^{\mathbb{N}} \leftrightarrows \mathcal{A}^{|\mathbb{N}|} : P \quad \text{where}$$
$$V((A_n, d_n)_n) := (A_n)_n \quad \text{and} \quad P((A_n)_n) := (\prod_{i=0}^{n} A_i, \pi_n).$$

Here $\pi_n : \prod_{i=0}^{n} A_i \to \prod_{i=0}^{n-1} A_i$ denotes the canonical projection in the first $n-1$ entries. Again it is left to the reader to show that this forms a pair of adjunction, that V is exact and that P preserves monomorphisms. Let $(A_n, d_n)_{n\in\mathbb{N}}$ be an element in $\mathcal{A}^{\mathbb{N}}$. Note that the morphism $(A_n, d_n)_{n\in\mathbb{N}} \to PV((A_n, d_n)_n)$, which is given by the unit of the adjunction, is a monomorphism. Now, choose an injective object $\mathcal{I} \in \mathcal{A}^{|\mathbb{N}|}$ toghether with a monomorphism $V((A_n, d_N)_n) \hookrightarrow \mathcal{I}$. Then by the above we get that

$$(A_n, d_n)_n \hookrightarrow PV((A_n, d_n)_n) \hookrightarrow P(\mathcal{I})$$

is a monomorphism into the injective object $P(\mathcal{I})$. This completes the proof of the first part. As we will not use the explicit description of injective objects in $\mathcal{A}^{\mathbb{N}}$, we defer the proof of the second statement to the literature [10, Proposition 1.1]. $\square$

Assume $\mathcal{A}$ is an abelian category which has small limits. The inverse limit functor

$$\lim_n : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}$$

is a left exact functor. If $\mathcal{A}$ has enough injectives, $\mathcal{A}^{\mathbb{N}}$ has enough injectives by Proposition 4.1.2. Therefore, we can right derive $\lim_n$ and define

$$\lim_n^p := R^p(\lim_n).$$

In the special case of $\mathcal{A} = \text{Ab}$ or Mod_R one can show the following two important statements.

Lemma 4.1.3. Let R be a ring. If $\mathcal{A} = \text{Ab}$ or Mod_R then for $p \geq 2$ we have

$$\lim_n^p = 0.$$

Proof. [14, Proposition 2.4.7] $\square$

[1] Given by families $(A_n)_{n\in\mathbb{N}}$ with all A_n injective.

Lemma 4.1.4. Assume $\mathcal{A} = \mathrm{Ab}$ or Mod_R and consider an inverse system $(F_n)_{n\in\mathbb{N}}$ which fulfills the Mittag-Leffler condition. Then

$$\lim_n^1 F_n = 0.$$

Proof. [14, Proposition 2.4.7] □

As next step, we want to investigate the passage to the AR-category. We will see that injective resolutions in $\mathrm{AR}(\mathcal{A})$ can be computed using injective resolutions in $\mathcal{A}^{\mathbb{N}}$. This, of course, is useful to determine the right derived of a functor, as one can simply use the well-known injectives in $\mathcal{A}^{\mathbb{N}}$. The crutial proposition is the following.

Proposition 4.1.5. If $\mathcal{A}$ has enough injectives then the AR-category of inverse systems $\mathrm{AR}(\mathcal{A})$ has enough injectives. Moreover, if $\mathcal{I} \in \mathcal{A}^{\mathbb{N}}$ is injective, then the induced object in the AR-category is injective.

Proof. Let F, G be two inverse systems and $f : F \overset{AR}{\hookrightarrow} G$ an AR-injective map. We will use Lemma 3.1.7 and explicitly work with the morphisms in $\mathrm{AR}(\mathcal{A})$. The map f is realized via a morphism $f' : F[r] \to G$ of inverse systems for an $r \in \mathbb{N}$. This induces an injective map of inverse systems $\tilde{f} : F[r]/\ker(f') \hookrightarrow G$. Define $H := F[r]/\ker(f')$ and note that there is an AR-isomorphism $H \cong_{\mathrm{AR}} F$.

Assume $F \to \mathcal{I}$ is an AR-morphism and $\mathcal{I}$ is injective in the category $\mathcal{A}^{\mathbb{N}}$. The composite

$$H \cong F \overset{AR}{\to} \mathcal{I}$$

is again realized via a morphism $H[r'] \to \mathcal{I}$ of inverse systems. As the shift $[r']$ is exact we get a diagram of inverse systems.

$$\begin{tikzcd} H[r'] \arrow[r, hook] \arrow[d] & G[r'] \arrow[dl, dotted] \\ \mathcal{I} & \end{tikzcd}$$

where the dotted arrow exists because $\mathcal{I}$ is injective. This diagram induces a commutative diagram in the AR-category

$$\begin{tikzcd} F \cong H \arrow[r, hook, "AR"] \arrow[d, "AR"'] & G \arrow[dl, dotted, "AR"] \\ \mathcal{I} & \end{tikzcd}.$$

This proves that $\mathcal{I}$ is injective in the AR-category. □

Definition 4.1.6. Recall that the inverse limit is a well-defined left exact functor

$$\begin{aligned} \lim_{\mathrm{AR}} : \mathrm{AR}(\mathcal{A}) &\longrightarrow \mathcal{A} \\ (F_n)_{n\in\mathbb{N}} &\longmapsto \lim_n F_n. \end{aligned}$$

As $\mathrm{AR}(\mathcal{A})$ has enough injectives if $\mathcal{A}$ has, we can consider its right derived $\mathrm{Rlim}_{\mathrm{AR}}$.

Corollary 4.1.7. Let $(F_n)_n$ be an inverse system in $\mathcal{A}$. Then

$$\mathrm{Rlim}_{\mathrm{AR}}(F_n)_n \cong \mathrm{Rlim}(F_n)_n.$$

Proof. An injective resolution in the AR-category can be computed via an injective resolution in $\mathcal{A}^{\mathbb{N}}$ by Proposition 4.1.5. A straightforward calculation yields the claim. □

4.2 Some Properties of Topoi

In the study of a site $\mathcal{C}$, it is essential to investigate the associated category of sheaves of sets, $\mathrm{Sh}(\mathcal{C})$. Many properties of a site translate to corresponding properties of $\mathrm{Sh}(\mathcal{C})$ and vice versa. This section briefly introduces the notion of a topos and discusses the properties of topoi that will be of certain interest for the pro-étale site. While the material in this section is standard, for an initial reading, we recommend that the reader focuses at least on the parts concerning weakly contractible objects and replete topoi. The material of this section can be found in [15, Tag 00X9] and originally in [1, Exposé IV].

Definition 4.2.1. A *topos* is the category $\mathrm{Sh}(\mathcal{C})$ of sheaves of sets on a site $\mathcal{C}$. A morphism of topoi $\mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$ is a tupel (f_*, f^{-1}), where

$$f_* : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C}) \quad \text{and} \quad f^{-1} : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$$

are functors such that the following hold.

- There is an isomorphism $\mathrm{Hom}_{\mathrm{Sh}(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$ functorial in $\mathcal{F}$ and $\mathcal{G}$.
- f^{-1} is left exact, i.e. commutes with finite limits.

Composition of morphism of topoi is carried out by composing the respective components of the tupel. Depending on the context, we sometimes write f^* for f^{-1}.

Remark 4.2.2. Let $\epsilon : \mathcal{C} \to \mathcal{D}$ be a morphism of sites. We can define a morphism of topoi $f : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ by setting $f_* := \epsilon_*$ and $f^{-1} := \epsilon^*$.

4.2.1 Representable Sheaves and Extension by Zero

Definition 4.2.3. Define the *Yoneda embedding* to be the functor

$$\begin{aligned} \mathrm{y} : \mathcal{C} &\longrightarrow \mathrm{PSh}(\mathcal{C}, \mathrm{Set}) \\ U &\longmapsto \mathrm{Mor}_{\mathcal{C}}(-, U). \end{aligned}$$

Presheaves of the form $\mathrm{y}(U)$ are called *representable presheaves*. Similarly, we define *representable sheaves* as sheaves of the form $\mathrm{ay}(U)$ for a $U \in \mathcal{C}$. We will also use the standard notation

$$h_U := \mathrm{y}(U) \quad \text{and} \quad h_U^{\#} := \mathrm{ay}(U).$$

Definition 4.2.4. A site $\mathcal{C}$ is called *subcanonical* if every representable presheaf is already a sheaf. The canonical topology from Example 1.1.4 is by definition the finest possible subcanonical topology.

We will make use of the following version of the Yoneda lemma.

Lemma 4.2.5. (Yoneda lemma) Let F be a presheaf on $\mathcal{C}$. Then for any $U \in \mathcal{C}$

$$\mathrm{Hom}_{\mathrm{PSh}}(h_U, F) \cong F(U).$$

In particular, if F is a sheaf then the universal property of sheafification implies $\mathrm{Hom}_{\mathrm{Sh}}(h_U^{\#}, F) \cong F(U)$. Restriction to V/U on the left handside is given by the precomposition with the canonical morphism $h_V \to h_U$.

Proof. [15, Tag 001P] □

Lemma 4.2.6. Let $\mathcal{C}$ be a site and $\{U_i \to U\}_{i \in I}$ a covering of $U \in \mathcal{C}$. Then the canonical morphism

$$\varphi : \bigsqcup_{i \in I} \mathrm{ay}(U_i) \to \mathrm{ay}(U)$$

of sheaves of sets is an epimorphism.

Proof. Let F be any sheaf and consider $s \in \mathrm{Hom}(\mathrm{ay}(U), F)$. Then s is the same as a section in $F(U)$. Consider the morphism

$$s \circ \varphi \in \mathrm{Hom}(\bigsqcup_{i \in I} \mathrm{ay}(U_i), F) = \prod_{i \in I} \mathrm{Hom}(\mathrm{ay}(U_i), F).$$

The morphism $s \circ \varphi$ is therefore the same as a family of sections $(s_i)_{i \in I}$, such that $s|_{U_i} = s_i$. The injectivity of

$$\mathrm{Hom}(\mathrm{ay}(U), F) \to \mathrm{Hom}(\bigsqcup_{i \in I} \mathrm{ay}(U_i), F)$$

follows now directly from the sheaf property of F. This implies that φ is an epimorphism and proves the claim. □

An important tool in topos theory is the *extension by zero*. For any $U \in \mathcal{C}$ and any abelian sheaf $F \in \mathrm{Ab}(\mathcal{C}/U)$ we will construct a sheaf $j_{U!}F \in \mathrm{Ab}(\mathcal{C})$. The functor $j_{U!}$ is a left adjoint of the restriction to U and has some useful properties.

Proposition 4.2.7. Let $\mathcal{C}$ be a site and $U \in \mathcal{C}$. The restriction functor

$$(-)|_U = j_U^* : \mathrm{Sh}(\mathcal{C}, \mathrm{Ab}) \longrightarrow \mathrm{Sh}(\mathcal{C}/U, \mathrm{Ab})$$

admits a left adjoint $j_{U!}$, called *the extension by zero*. Moreover, if $\mathcal{G}$ is an abelian sheaf, then the extension by zero $j_{U!}\mathcal{G}$ is given by the sheafification of

$$V \longmapsto \bigoplus_{\varphi \in \mathrm{Hom}_{\mathcal{C}}(V,U)} \mathcal{G}(\varphi : V \to U).$$

Similarly, the restriction functor $(-)|_U = j_U^* : \mathrm{Sh}(\mathcal{C}, \mathrm{Set}) \longrightarrow \mathrm{Sh}(\mathcal{C}/U, \mathrm{Set})$ on the sheaves of sets has a left adjoint which we will also denote by $j_{U!}$. It is called *extension by the empty set* and for a sheaf of sets $\mathcal{F}$ it is given by the sheafification of

$$V \longmapsto \bigsqcup_{\varphi \in \mathrm{Hom}_{\mathcal{C}}(V,U)} \mathcal{G}(\varphi : V \to U).$$

Proof. The statement concerning abelian sheaves is [15, Tag 03DI] applied to $\mathcal{O} = \mathrm{a}\underline{\mathbb{Z}}$. The claim about sheaves of sets is proved in [15, Tag 00XZ]. For a more general statement defining the extension of zero for morphisms of topoi with certain properties see [15, Tag 09YW]. □

Lemma 4.2.8. The extension by zero $j_{U!}$ is an exact functor $\mathrm{Sh}(\mathcal{C}/U, \mathrm{Ab}) \longrightarrow \mathrm{Sh}(\mathcal{C}, \mathrm{Ab})$.

Proof. As $j_{U!}$ is a left adjoint, it is right exact by general knowledge. So it suffices to show that any injection of sheaves $\mathcal{F} \hookrightarrow \mathcal{G}$ on $\mathcal{C}/U$ turns into an injection $j_{U!}\mathcal{F} \to j_{U!}\mathcal{G}$. Let $V \in \mathcal{C}$ and consider the induced morphism of presheaves

$$\bigoplus_{\varphi \in \mathrm{Hom}_{\mathcal{C}}(V,U)} \mathcal{F}(\varphi : V \to U) \longrightarrow \bigoplus_{\varphi \in \mathrm{Hom}_{\mathcal{C}}(V,U)} \mathcal{G}(\varphi : V \to U).$$

It is injective, because $\mathcal{F} \hookrightarrow \mathcal{G}$ is injective and taking sections is left exact. As sheafification is exact, the map between the associated sheaves is injective. This map is exactly $j_{U!}\mathcal{F} \to j_{U!}\mathcal{G}$, which proves the claim. □

Lemma 4.2.9. The extension by the empty set preserves monomorphisms and epimorphisms.

Proof. The main idea is to observe that the extension by the empty set gives rise to an equivalence of categories

$$\mathrm{Sh}(\mathcal{C}/U, \mathrm{Set}) \xrightarrow{\cong} \mathrm{Sh}(\mathcal{C}, \mathrm{Set})/\mathrm{ay}(U).$$

A justification about this equivalence can be found in [15, Tag 00Y1]. Now, the statement becomes clear as the canonical functor $\mathrm{Sh}(\mathcal{C}, \mathrm{Set})/\mathrm{ay}(U) \to \mathrm{Sh}(\mathcal{C}, \mathrm{Set})$ obviously preserves monomorphisms and epimorphisms. □

Lemma 4.2.10. Let $\mathcal{C}$ be a site and $X \in \mathcal{C}$ the terminal object. Assume $U \to X$ is a monomorphism in $\mathcal{C}$, then the canonical map

$$\mathcal{F} \to j_U^* j_{U!} \mathcal{F}$$

is an isomorphism. This is true both for abelian sheaves and for sheaves of sets.

Proof. We restrict to the case of abelian sheaves. The proof for sheaves of sets works analogously. Let V be an object in $\mathcal{C}$. The assumption that $U \hookrightarrow X$ is a monomorphism directly translates to the fact that there is at most one morphism from V to U in the site $\mathcal{C}$. In particular, $j_{U!}$ for a sheaf $\mathcal{F}$ on $\mathcal{C}/U$ is given by the sheafification of

$$\tilde{\mathcal{F}} : V \longmapsto \begin{cases} 0 & \text{if there is no morphism } V \to U, \\ \mathcal{F}(V \to U) & \text{else.} \end{cases}$$

It is elementary to show that restriction commutes with sheafification[2]. Hence, one obtains the formula $(j_{U!}\mathcal{F})|_U = \mathrm{a}(\tilde{\mathcal{F}}|_U)$. But $\tilde{\mathcal{F}}|_U$ equals $\mathcal{F}$ and, therefore, sheafification is redundant. This finishes the proof. □

4.2.2 Open and Closed Subtopoi

Definition 4.2.11. A morphism of topoi $(f_*, f^{-1}) : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$ such that f_* is fully faithful is called *embedding*. A strictly full subcategory[3] $E \subset \mathrm{Sh}(\mathcal{C})$ is called *subtopos* if it is the essential image of f_* for an embedding (f_*, f^{-1}).

Lemma 4.2.12. Let $\mathcal{C}$ be a site and $U \in \mathcal{C}$. Then the extension by zero gives rise to an equivalence of categories

$$j_{U!} : \mathrm{Sh}(\mathcal{C}/U) \xrightarrow{\cong} \mathrm{Sh}(\mathcal{C})/h_U^\#$$

Proof. [15, Tag 00Y1] □

Lemma 4.2.13. Let X be the terminal object in $\mathcal{C}$. If $U \hookrightarrow X$ is a monomorphism, then $h_U^\#$ is a subsheaf of the terminal object of $\mathrm{Sh}(\mathcal{C})$.

Proof. The assumptions imply that the representable presheaf h_U is a subpresheaf of h_X. As sheafification preserves injections, $h_U^\#$ is a subsheaf of $h_X^\#$. The latter is exactly the terminal object of $\mathrm{Sh}(\mathcal{C})$. □

Lemma 4.2.14. Assume $\epsilon : \mathcal{C} \to \mathcal{D}$ is a morphism of sites. Then the functor $\epsilon^{-1} : \mathcal{D} \to \mathcal{C}$ preserves monomorphisms.

[2] A possible proof uses that both, the restriction and the sheafification, are left adjoints.
[3] I.e. E is full and if $A \cong B$ in $\mathrm{Sh}(\mathcal{C})$ and $A \in E$ then $B \in E$.

Proof. Let $V \hookrightarrow U$ be a monomorphism in $\mathcal{D}$. Equivalently, the diagonal morphism $V \to V \times_U V$ is an isomorphism. As ϵ^{-1} preserves pullbacks by definition, the statement follows immediately. $\square$

Lemma 4.2.15. For an $\mathcal{F} \in \mathrm{Sh}(\mathcal{C})$ the following are equivalent:

- $\mathrm{Sh}(\mathcal{C})/\mathcal{F}$ is a subtopos of $\mathrm{Sh}(\mathcal{C})$.
- $\mathcal{F}$ is a subsheaf of the terminal object of $\mathrm{Sh}(\mathcal{C})$.

Proof. [15, Tag 08LW] $\square$

Lemma 4.2.16. Let $\mathcal{F}$ be a subsheaf of the terminal object of $\mathrm{Sh}(\mathcal{C})$. Then the category of all sheaves $\mathcal{G} \in \mathrm{Sh}(\mathcal{C})$ such that

$$\mathcal{F} \times \mathcal{G} \xrightarrow{\cong} \mathcal{F}$$

is an isomorphism, forms a subtopos of $\mathrm{Sh}(\mathcal{C})$.

Proof. [15, Tag 08LY] $\square$

Definition 4.2.17. An *open subtopos* of $\mathrm{Sh}(\mathcal{F})$ is a subtopos of the form $\mathrm{Sh}(\mathcal{C})/\mathcal{F}$ where $\mathcal{F}$ is a subsheaf of the terminal object of $\mathrm{Sh}(\mathcal{C})$. The *complementary closed subtopos* of $\mathrm{Sh}(\mathcal{C})/\mathcal{F}$ is then defined to be the subtopos from Lemma 4.2.16. Similary, one uses Lemma 4.2.13 to define the associated open or closed subtopoi for monomorphism $U \hookrightarrow X$ into the terminal object X of $\mathcal{C}$.

Example 4.2.18. Let X be a scheme and $i : Z \hookrightarrow X$ a closed subscheme. Let $j : U \hookrightarrow X$ be the open subscheme given by the complement of Z. It is easy to show that h_U is a subsheaf of the terminal object of $\mathrm{Sh}(X_{\mathrm{et}})$. Together with the formula $U_{\mathrm{et}} \cong X_{\mathrm{et}}/U$, this implies that $\mathrm{Sh}(U_{\mathrm{et}})$ is an open subtopos of $\mathrm{Sh}(X_{\mathrm{et}})$. Now consider the pushforward i_* of the closed immersion. It is fully faithful as

$$\mathrm{Hom}_{\mathrm{Sh}(X_{\mathrm{et}})}(i_*\mathcal{F}, i_*\mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Z_{\mathrm{et}})}(\mathcal{F}, i^*i_*\mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}(Z_{\mathrm{et}})}(\mathcal{F}, \mathcal{G}).$$

Further, for any sheaf $\mathcal{G} \in \mathrm{Sh}(Z_{\mathrm{et}})$ and any étale scheme U' over X we have

$$i_*\mathcal{G} \times h_U(U') = \mathcal{G}(i^{-1}(U')) \times h_U(U') = \begin{cases} \mathcal{G}(\emptyset) \times \{*\} = \{*\} & \text{if } U' \text{ factorizes over } U, \\ \emptyset & \text{else.} \end{cases}$$

It is elementary to show that this is canonically isomorphic to h_U. Conversely, assume $\mathcal{F}$ is a sheaf on X_{et} such that $\mathcal{F} \times h_U \to h_U$ is an isomorphism. It follows that $\mathcal{F}|_U$ is the sheaf with $\mathcal{F}|_U(U') = \emptyset$ for all $U' \in X_{\mathrm{et}}/U$. A stalkwise calculation[4] shows that $\mathcal{F} \cong i_*i^*\mathcal{F}$. Hence, the topos $\mathrm{Sh}(Z_{\mathrm{et}})$ is actually the complementary closed subtopos of $\mathrm{Sh}(U_{\mathrm{et}})$.

[4]For any $x \in Z$ we have $i_*i^*\mathcal{F}_x = \mathcal{F}_x$ and for $x \in U$ we have $i_*i^*\mathcal{F}_x = \{*\} = \mathcal{F}_x$.

For the following fix a topos $\mathcal{T}$ and an open subtopos $\mathcal{U} \subset \mathcal{T}$. Let $\mathcal{A}$ be the complementary closed subtopos of $\mathcal{U}$ in $\mathcal{T}$ and let R be a sheaf of rings. Further, let

$$j : \mathcal{U} \to \mathcal{T} \quad \text{and} \quad i : \mathcal{A} \to \mathcal{T}$$

be the canonical morphisms of topoi. We then formulate the following proposition.

Proposition 4.2.19. For any sheaf of R-modules $\mathcal{F}$ there is a canonical short exact sequence

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0 \tag{4.1}$$

functorial in $\mathcal{F}$.

Proof. [1, p. 269] □

Proposition 4.2.20. The functor $i_* : \mathrm{Mod}_R(\mathcal{A}) \to \mathrm{Mod}_R(\mathcal{T})$ has a right adjoint $i^!$. Moreover, the counit $i_* i^! \to \mathrm{id}$ is an isomorphism and any sheaf of R-modules $\mathcal{F}$ fits into a left exact sequence

$$0 \to i^! i_* \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}$$

Proof. [1, Proposition 14.5] and [1, Proposition 14.6]. □

Corollary 4.2.21. The functor i_* commutes with small limits and colimits. In particular, i_* is exact.

4.2.3 Weakly Contractible Objects and Replete Topoi

For this subsection, let $\mathcal{C}$ be a site.

Definition 4.2.22. An object $U \in \mathcal{C}$ is called *weakly contractible* if for any epimorphism $\mathcal{F} \to \mathcal{G}$ of sheaves of sets the induced map

$$\mathcal{F}(U) \to \mathcal{G}(U)$$

is surjective. We say $\mathcal{C}$ has *enough weakly contractible objects* if every $V \in \mathcal{C}$ has a covering $(U_i \to V)_{i \in I} \in \mathrm{Cov}(V)$ by weakly contractible objects U_i.

Definition 4.2.23. Let $\mathcal{C}$ be a site and $\mathcal{T} := \mathrm{Sh}(\mathcal{C})$ the induced topos. An object $F \in \mathcal{T}$ is called *weakly contractible* if any surjection

$$\pi : G \twoheadrightarrow F$$

has a section $s : F \to G$, i.e. $\pi \circ s = \mathrm{id}_F$. The topos $\mathcal{T}$ is called *locally weakly contractible* if for any $G \in \mathcal{T}$ there is a family $(F_i)_{i \in I}$ of weakly contractible objects in $\mathcal{T}$ and a surjection

$$\coprod_{i \in I} F_i \twoheadrightarrow G.$$

Although Definition 4.2.22 and 4.2.23 look different, we will see that any site $\mathcal{C}$ with enough weakly contractible objects produces a locally weakly contractible topos $\mathrm{Sh}(\mathcal{C})$. In fact, any weakly contractible object gives rise to a weakly contractible object in $\mathrm{Sh}(\mathcal{C})$.

Proposition 4.2.24. The following statements are equivalent for $U \in \mathcal{C}$:

1. U is weakly contractible.
2. $\mathrm{ay}(U)$ is a weakly contractible object in the topos $\mathrm{Sh}(\mathcal{C})$.
3. For every covering $(U_i \to U)_{i \in I}$ there is a section $s : \mathrm{ay}(U) \to \bigsqcup_i \mathrm{ay}(U_i)$ for the canonical morphism of sheaves
$$t : \bigsqcup_i \mathrm{ay}(U_i) \to \mathrm{ay}(U).$$

Proof. We first show that 1 implies 2. Assume there is a surjection $t : F \twoheadrightarrow \mathrm{ay}(U)$. Then, as U is weakly contractible, $\mathrm{id}_U \in \mathrm{ay}(U)(U)$ has a preimage $s \in F(U)$. By the Yoneda lemma, s is the same as a morphism $s : \mathrm{ay}(U) \to F$ with $t \circ s = \mathrm{id}$, which defines a section for t.
Now assume $\mathrm{ay}(U)$ is a weakly contractible object in the topos $\mathrm{Sh}(\mathcal{C})$ and let $(U_i \to U)_{i \in I}$ be a covering of U. By Lemma 4.2.6 we get a surjection
$$\varphi : \bigsqcup_i \mathrm{ay}(U_i) \longrightarrow \mathrm{ay}(U).$$
Now, the assumptions immediately imply 3.
Assume we have a surjection of sheaves $\varphi : F \to G$ and $U \in \mathcal{C}$ fulfilling 3. Let $g \in G(U)$ be a section. We can find a covering $\{U_i \to U\}_i$ of U and elements $f_i \in F(U_i)$ such that f_i is mapped to $g|_{U_i}$ under φ. By assumption, the canonical epimorphism $t : \bigsqcup_i \mathrm{ay}(U_i) \to \mathrm{ay}(U)$ has a right inverse
$$s : \mathrm{ay}(U) \to \bigsqcup_i \mathrm{ay}(U_i).$$
Using the Yoneda Lemma, one sees that the elements f_i are the same as morphism $f_i : \mathrm{ay}(U_i) \to F$. Now define an element $f \in F(U)$ via the morphism
$$f : \mathrm{ay}(U) \xrightarrow{s} \bigsqcup_i \mathrm{ay}(U_i) \xrightarrow{\sqcup_i f_i} F.$$
We claim that f is mapped to g via φ. This can be seen considering the commutative diagram
$$\begin{tikzcd}
\mathrm{ay}(U) \arrow[d, "s"] \arrow[dr, bend left, "f"] \arrow[dd, bend right=60, dotted, "\mathrm{id}"'] & \\
\bigsqcup_i \mathrm{ay}(U_i) \arrow[r, "\sqcup_i f_i"] \arrow[d, "t"] & F \arrow[d, "\varphi"] \\
\mathrm{ay}(U) \arrow[r, "g"] & G.
\end{tikzcd}$$

We showed that $\varphi \circ f = g \in G(U)$ which completes the proof. □

Corollary 4.2.25. Assume $\mathcal{C}$ has enough weakly contractible objects. Then the topos $\mathrm{Sh}(\mathcal{C})$ is locally weakly contractible.

Remark 4.2.26. The literature sometimes uses the following definition of weakly contractible objects. An object $U \in \mathcal{C}$ is weakly contractible if every covering morphism $V \to U$ has a section. It is not clear if this definition agrees with our definition in general, but it is at least true if the site $\mathcal{C}$ has certain "nice" properties. More precisely, M. Kerz showed in [11] that in an admissible site $\mathcal{C}$ the following is equivalent for $U \in \mathcal{C}$:

- Every covering morphism $V \to U$ has a section.
- Every surjection of sheaves $\mathcal{F} \to \mathrm{y}(U)$ has a section.

But in fact, the latter is exacly one of the equivalent statements given in Proposition 4.2.24.

Lemma 4.2.27. Let $U \in \mathcal{C}$ be a weakly contractible object. Then the sections functor $\Gamma(U,-) : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}$ is exact.

Proof. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of abelian sheaves then we get a left exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U).$$

The surjectivity of $\mathcal{G}(U) \to \mathcal{H}(U)$ follows from the definition of weakly contractible objects. □

Proposition 4.2.28. Assume $\mathcal{C}$ has enough weakly contractible objects. Let $\mathcal{B} \subset \mathcal{C}$ be a subcategory such that every object in $\mathcal{B}$ is weakly contractible and every element in $\mathcal{C}$ admits a covering by elements in $\mathcal{B}$. Let $\mathcal{O}$ be a sheaf of rings. Then a sequence of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact if and only if

$$\mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact for every $U \in \mathcal{B}$. In particular, checking if a map $\mathcal{F} \to \mathcal{G}$ is an isomorphism can be checked on sections of weakly contractible objects in $\mathcal{B}$.

Proof. This is clear using the defining property of weakly contractible objects and the assumption that every element $U \in \mathcal{C}$ can be covered by weakly contractible objects. □

Lemma 4.2.29. Let $\mathcal{F}$ be a sheaf of Λ-modules on a site $\mathcal{C}$ and let $U \in \mathcal{C}$ be a weakly contractible object. For any finitely generated ideal $I \subset \Lambda$ we have

$$\Gamma(U, I\mathcal{F}) = I\Gamma(U, \mathcal{F}).$$

Proof. Write $I = (a_1, \dots, a_n)$ for elements $a_i \in \Lambda$. Then we see

$$I\mathcal{F} = \mathrm{im}\left(\underline{I} \otimes_{\underline{\Lambda}} \mathcal{F} \to \mathcal{F}\right) = \mathrm{im}\left(\left(\underline{(a_1)} \otimes_{\underline{\Lambda}} \mathcal{F}\right) \oplus \dots \oplus \left(\underline{(a_n)} \otimes_{\underline{\Lambda}} \mathcal{F}\right) \to \mathcal{F}\right).$$

As taking section at U is exact, it commutes with images and direct sums. This is

$$\Gamma(U, I\mathcal{F}) = \mathrm{im}\left(\Gamma\left(U, \underline{(a_1)} \otimes_{\underline{\Lambda}} \mathcal{F}\right) \oplus \dots \oplus \Gamma\left(U, \underline{(a_n)} \otimes_{\underline{\Lambda}} \mathcal{F}\right) \to \Gamma(U, \mathcal{F})\right). \tag{4.2}$$

For $i \in \{1, \dots, n\}$ we can compute $\mathrm{im}\left(\Gamma\left(\underline{(a_i)} \otimes_{\underline{\Lambda}} \mathcal{F}, U\right) \to \Gamma(\mathcal{F}, U)\right)$. Consider the factorization

$$\begin{tikzcd}
\underline{(a_i)} \otimes_{\underline{\Lambda}} \mathcal{F} \arrow[r] & \mathcal{F} \\
\mathcal{F} \arrow[u, "(\cdot a_i)\otimes \mathrm{id}"] \arrow[ur, "\cdot a_i"'] &
\end{tikzcd}$$

This shows $\mathrm{im}\left(\Gamma\left(\underline{(a_i)} \otimes_{\underline{\Lambda}} \mathcal{F}, U\right) \to \Gamma(\mathcal{F}, U)\right) = a_i \Gamma(\mathcal{F}, U)$. Together with Eq. (4.2) this proves the claim. □

Example 4.2.30. Having enough weakly contractible objects is a property that, in general, the sites defined in this thesis do not possess:

- Consider the Zariski site X_{Zar} defined by the scheme $X := \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[T])$. Then X_{Zar} has no weakly contractible objects except for the empty subscheme. To proof this, consider an open non-empty subscheme $U \subset X$. Then by basic knowlege, U contans infinitely many closed points. Choose two distinct closed points $p, q \in U$ and define $Y := \{p, q\}$ and $V := X \setminus Y$. We have a closed immersion $i : Y \hookrightarrow X$. We denote $\mathbb{Z}_Y := i_* i^* \underline{\mathbb{Z}}$ as the extension of zero of the constant sheaf $\underline{\mathbb{Z}}$. The unit of the adjunction (i^*, i_*) induces a morphism $\eta : \underline{\mathbb{Z}} \to \mathbb{Z}_Y$. One computes on stalks

$$\underline{\mathbb{Z}}_x = \mathbb{Z} \quad \text{and} \quad \mathbb{Z}_{Y,x} = \begin{cases} \mathbb{Z} & \text{if } x \in \{p, q\}, \\ 0 & \text{else.} \end{cases}$$

  It is now easy to show that η is an epimorphism. Using the sheaf property we can compute

$$\mathbb{Z}_Y(U) \cong \{(x, y) \in \mathbb{Z}_Y(U \setminus \{p\}) \times \mathbb{Z}_Y(U \setminus \{q\}) \mid x|_{U \setminus Y} = y|_{U \setminus Y}\}.$$

  As U is irreducible, $\mathbb{Z}_Y|_{U \setminus Y} = 0$ and $\mathbb{Z}_Y(U \setminus \{p\}) = \mathbb{Z}_Y(U \setminus \{q\}) = \mathbb{Z}$ and we immediately get the formula

$$\underline{\mathbb{Z}}(U) = \mathbb{Z} \quad \text{and} \quad \mathbb{Z}_Y(U) = \mathbb{Z} \times \mathbb{Z}.$$

  Further $\eta(U)$ is the morphism sending 1 to $(1, 1)$ and is therefore not surjective.

- The étale site associated to a scheme has in general not enough weakly contractible objects, see Examples 4.2.32.

- If X is any scheme then the pro-étale site X_{proet} will have enough weakly contractible objects.

Before stating the main axioms, introduce an interesting property of a site with enough weakly contractible objects. In connection with this a "good" behavior of limits arises. In the following sections, we will often make use of this property. For instance, it is used to derive a connection between the cohomology of the pro-étale site and Uwe Jannsens coninuous étale cohomology [10].

Definition 4.2.31. A topos $\mathcal{T}$ is *replete* if for any sequence of morphisms

$$F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow \cdots$$

with surjective transition maps the induced morphism $\lim_i F_i \to F_n$ is surjective for any $n \in \mathbb{N}$.

Examples 4.2.32.

- The category Set of sets is replete.
- For any site $\mathcal{C}$ the topos $\mathrm{PSh}(\mathcal{C})$ is replete.
- The following example from [2, Example 3.1.5] shows that the topos induced by the étale site is in general not replete. Let k be a field such that a separable closure of k is not finite over k. Let $\overline{k}$ be a separable closure. Then the topos $\mathrm{Sh}(\mathrm{Spec}(k)_{\text{et}})$ is not replete. To see this consider a chain

  $$k = k_1 \subset k_2 \subset \cdots \subset \overline{k}$$

  of non-trivial finite separable field extensions $k \subset k_i$. We then get a sequence of representable sheaves

  $$h_{\mathrm{Spec}(k_1)} \leftarrow h_{\mathrm{Spec}(k_2)} \leftarrow \cdots \leftarrow h_{\mathrm{Spec}(k_n)}.$$

  The transition maps are surjective as they come from the covering morphisms $\mathrm{Spec}(k_{i+1}) \to \mathrm{Spec}(k_i)$. We claim that the canonical map $\lim_i h_{\mathrm{Spec}(k_i)} \to h_{\mathrm{Spec}(k)}$ is not surjective. Any étale covering of $\mathrm{Spec}(k)$ can be refined by a covering of the form $(\mathrm{Spec}(L_i) \to \mathrm{Spec}(k))_i$ with $k \subset L_i$ a finite separable field extension. Consider the element $\mathrm{id}_k \in h_{\mathrm{Spec}(k)}(\mathrm{Spec}(k))$ and assume there is a finite separable field extension $k \subset L$ such that $\mathrm{id}_k|_{\mathrm{Spec}(L)}$ is in the image of

  $$\lim_i h_{\mathrm{Spec}(k_i)}(\mathrm{Spec}(L)) \to h_{\mathrm{Spec}(k)}(\mathrm{Spec}(L))$$

  But this exactly means that for any $i \in \mathbb{N}$ there is a k-morphism $k_i \to L$. This is a contradiction as k-morphisms of fields are injective k-linear mappings, L is finite over k and $\dim_k(k_i) \geq i$ for all $i \in \mathbb{N}$. Hence, the morphism $\lim_i h_{\mathrm{Spec}(k_i)} \to h_{\mathrm{Spec}(k)}$ cannot be surjective, which proves the claim.

Proposition 4.2.33. Let $\mathcal{C}$ be a site with enough weakly contractible objects. Then the topos $\mathrm{Sh}(\mathcal{C})$ is replete.

Proof. Consider a system $F : \mathbb{N}^{\mathrm{opp}} \to \mathrm{Sh}(\mathcal{C})$ with surjective transition maps. To show that

$$\lim_n F_n \to F_i$$

is surjective, it suffices to show that it surjective after applying $\Gamma(U, -)$ for arbitrary weakly contractibles U. This is obviously true as Set is a replete topos and the system $(F_i(U))_{i\in\mathbb{N}}$ has surjective transition maps. □

Proposition 4.2.34. Let $\mathcal{T} = \mathrm{Sh}(\mathcal{C})$ be a replete topos and $F : \mathbb{N}^{\mathrm{opp}} \to \mathrm{Ab}(\mathcal{C})$ be a diagram with surjective transition maps $F_{i+1} \to F_i$. Then

$$\mathrm{Rlim} F_n \cong \lim_n F_n.$$

Proof. [2, Proposition 3.1.10] □

4.3 Axioms for a Pro-étale Enlargement

This section contains the promised axioms for the pro-étale site. After introducing the aforementioned axioms, we will provide an explanation of our intention behind their formulation. This is followed by some basic definitions and first statements which then lead to the definition of adic pro-étale sheaves in Section 4.4. Note, that the axioms are designed such that they yield a good theory of adic pro-étale sheaves and that it is not a standard notation in the literature.

Definition 4.3.1 (Axioms for the pro-étale site)**.** Let $E \subset \mathrm{Sch}$ be a subcategory. A *pro-étale enlargement for E* is a functor

$$(-)_{\mathrm{proet}} : E \to \mathrm{Sites}$$

together with a natural transformation

$$\epsilon : (-)_{\mathrm{proet}} \longrightarrow (-)_{\mathrm{et}}$$

such that for any scheme $X \in E$ the following axioms are fulfilled[5]. By abuse of notation, we write ϵ for the morphism of sites $\epsilon(X) : X_{\mathrm{proet}} \to X_{\mathrm{et}}$.

1. The site X_{proet} has enough weakly contractible objects.
2. For any abelian sheaf $\mathcal{F}$ on X_{et} we have

$$R^i\epsilon_*(\epsilon^*\mathcal{F}) = \begin{cases} \mathcal{F} & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$$

[5]Note that we are ignoring problems that arise with the notion of 2-categories, see Remark 2.1.3.

3. Let $j : U \to X$ be a morphism in E. If $j : U \hookrightarrow X$ is an open subscheme then we have a commutative diagram

$$\begin{tikzcd} X_{\text{proet}}/\epsilon^{-1}(U) \arrow[r] \arrow[d, "\cong"'] & X_{\text{proet}} \arrow[d, "\text{id}"] \\ U_{\text{proet}} \arrow[r, "j_{\text{proet}}"] & X_{\text{proet}}. \end{tikzcd}$$

   Here, the morphism of sites $X_{\text{proet}}/\epsilon^{-1}(U) \to X_{\text{proet}}$ is the restriction morphism defined in Section 1.1.3. In particular, $\text{Sh}(U_{\text{proet}})$ defines an open subtopos of $\text{Sh}(X_{\text{proet}})$.

4. Let $i : Z \to X$ be a morphism in E. Assume $i : Z \hookrightarrow X$ is a closed immersion such that its open complement $j : U \hookrightarrow X$ is quasi-compact and lies in E. Then $\text{Sh}(Z_{\text{proet}})$ is the complementary closed subtopos of $\text{Sh}(U_{\text{proet}})$. That means, $i_{\text{proet}*}$ is fully faithful and its essential image is the complementary closed subtopos of $\text{Sh}(U_{\text{proet}})$.

5. The pullback i^*_{proet} along any closed immersion $i : Z \hookrightarrow X$ in E has a left adjoint $i_!$. In particular, i^*_{proet} commutes with small limits.

Notation 4.3.2 (The meaning of $(*)$)**.** Let X be a scheme and let $\{X\}$ be the category with the single object X and only one morphism. Note that a pro-étale enlargement for $\{X\}$ exactly consists of a site X_{proet} together with a morphism $\epsilon : X_{\text{proet}} \to X_{\text{et}}$ such that the Axioms 1 and 2 are fulfilled. In the following we mark those theorems, propositions and lemmas with $(*)$ if the respective statement works for any pro-étale enlargement for $\{X\}$.

Remark 4.3.3 (about the axioms)**.** We want to explain the intention behind our choice of the axioms. In the literature there are two concrete approaches to define a pro-étale topology, one from Bhatt and Scholze [2] and another from M. Kerz [12]. Our goal was to collect the most important properties, which are true in both versions. The axioms are designed, such that the notion of adic sheaves on the pro-étale site, introduced in Section 4.4, comes with good properties and is in equivalence with the definition of classical adic sheaves. However, let us briefly go through the axioms and explain the underlying intentions behind each one.

The first axiom is definitely the most important one. The existence of weakly contractible objects is the main advantage that comes with the pro-étale site.

We need the second axiom to draw a connection between étale and pro-étale sheaves. We can apply this property for $i = 0$ and see that $\epsilon_*\epsilon^* \cong \text{id}$ and, therefore, ϵ^* is a fully faithful functor. Moreover, this axiom guarantees that the cohomology groups of an abelian sheaf $\mathcal{F}$ on X_{et} agree with the cohomology group of $\epsilon^*\mathcal{F}$ on X_{proet}, see Proposition 4.3.6. Of course, this is a property that the pro-étale site should have in order to extend the classical theory.

At first glance, Axiom 3 seems rather restricting because defining a pro-étale site associated to X requires careful consideration of the behavior of sites induced from any open subscheme $U \subset X$. However, this is not a significant problem since the first two axioms are automatically fulfilled for the localization category X_{proet}/U if they hold in X_{proet}. This statement is specified in Proposition 4.3.15. Moreover, without assuming this axiom, two different notions of restriction to open subschemes arise (see Section 1.1.3 and Definition 4.3.4).

The latter axiom describes a property of the pro-étale site for open subschemes. Of course, one also wants conditions for closed subschemes. Axiom 4 is the canonical one. It comes with many advantages, especially the short exact sequence (4.1) from Proposition 4.2.19. We had to restrict this axiom for immersions with quasi-compact open complement, as one needs this assumption in the pro-étale site of B. Bhatt and P. Scholze to show that the pushforward along a closed immersion is exact. Note that in their version it is not known if i_* for arbitrary closed subschemes $i : Z \hookrightarrow X$ is exact.

Finally, we introduced Axiom 5 to ensure that the extension by zero along a quasi-compact open immersion preserves adic constructible sheaves. In their paper [2], Bhatt and Scholze state that the existence of a left adjoint to i^* is an important property of the pro-étale site, which is in general not true for the étale site. As this axiom is valid in both versions, we decided that it is necessary to include it into the elementary properties of the pro-étale site.

In the following let $((-)_{\text{proet}}, \epsilon)$ be a pro-étale enlargement for $E = \text{Sch}$. Of course, the statements can be adjusted in a straightforward manner to work for more general subcategories $E \subset \text{Sch}$.

Definition 4.3.4. Let X be a scheme and $\iota : Z \hookrightarrow X$ a locally closed immersion. For a sheaf $\mathcal{F} \in \text{Sh}(X_{\text{et}})$ define the *restriction to Z* as

$$\mathcal{F}|_Z := \iota_{\text{proet}}^* \mathcal{F},$$

where $\iota_{\text{proet}} : Z_{\text{proet}} \to X_{\text{proet}}$ is the induced morphism of sites. By Axiom 3 the restriction to an open subscheme is exactly the restriction defined in Definition 1.1.25.
Let $f : X \to Y$ be a morphism of schemes. To simplify notation, we often write f for the morphism of sites f_{proet}.

Remark 4.3.5. Axiom 3 assures that for an open immersion $j : U \hookrightarrow X$, we have $U_{\text{proet}} = X_{\text{proet}}/\epsilon^{-1}(U)$. In particular, by the material of Section 4.2.1, the pullback j^* has a left adjoint $j_!$. To get a better insight in this theory, we recommend to recall Section 1.1.3 and 4.2.1.

Proposition 4.3.6 (∗)**.** Let F be an abelian sheaf on X_{et}. We then have the equality

$$H^i(X_{\text{et}}, F) = H^i(X_{\text{proet}}, \epsilon^* F)$$

Proof. This is a direct consequence of Axiom 2 and the Leray spectral sequence. For instance, see [15, Tag 0733]. □

Definition 4.3.7. Let X be a scheme, Λ a noetherian ring and $F \in \mathrm{Mod}_\Lambda(X_{\mathrm{proet}})$. Then F is called *constructible* if it is isomorphic to $\epsilon^* G$ for a constructible étale sheaf G.

Lemma 4.3.8 (∗)**.** The pullback ϵ^* gives rise to an equivalence of categories

$$\epsilon^* : \{\text{Constructible sheaves on } X_{\mathrm{et}}\} \xrightarrow{\cong} \{\text{Constructible sheaves on } X_{\mathrm{proet}}\}$$

Proof. By Axiom 2, ϵ^* is fully faithful. Now, the claim follows directly from the definitions. □

Proposition 4.3.9 (∗)**.** Let $\mathcal{S} \subset \mathrm{Ab}(X_{\mathrm{et}})$ be a weak Serre subcategory. Then the essential image of $\mathcal{S}$ in X_{proet} via the functor ϵ^* is a weak Serre subcategory of $\mathrm{Ab}(X_{\mathrm{proet}})$

Proof. Assume we have $\mathcal{F} = \epsilon^* F$, $\mathcal{H} = \epsilon^* H$ for sheaves $F, H \in \mathcal{S}$ and a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.$$

By Axiom 2 we have $R^1\epsilon_*\mathcal{F} = 0$, so we obtain an exact sequence

$$0 \to \underbrace{\epsilon_*\mathcal{F}}_{\cong F} \to \epsilon_*\mathcal{G} \to \underbrace{\epsilon_*\mathcal{H}}_{\cong H} \to 0.$$

By assumption $\epsilon_*\mathcal{G} \in \mathcal{S}$ and the 5-lemma applied to

$$\begin{tikzcd}
0 \arrow[r] & \epsilon^*\epsilon_*\mathcal{F} \arrow[r] \arrow[d, "\cong"'] & \epsilon^*\epsilon_*\mathcal{G} \arrow[r] \arrow[d] & \epsilon^*\epsilon_*\mathcal{H} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0
\end{tikzcd}$$

yields the claim. □

Corollary 4.3.10 (∗)**.** The category of constructible sheaves on X_{proet} forms a weak Serre subcategory of the category of sheaves on X_{proet}.

Lemma 4.3.11. Let $f : Y \to X$ be a morphism of schemes. Then we have the following properties

1. $f_{\mathrm{et}*} \circ \epsilon_* = \epsilon_* \circ f_{\mathrm{proet}*}$ and $f^*_{\mathrm{proet}} \circ \epsilon^* = \epsilon^* \circ f^*_{\mathrm{et}}$.

2. Assume f is a closed immersion with quasi-compact complement, then the formula $f_{\mathrm{proet}*} \circ \epsilon^* = \epsilon^* \circ f_{\mathrm{et}*}$ holds.

3. If f is an open immersion then $f^*_{\mathrm{et}}\epsilon_* F = \epsilon_* f^*_{\mathrm{proet}} F$ and $f_{\mathrm{proet}!}\epsilon^* F = \epsilon^* f_{\mathrm{et}!} F$.

Proof. The first claim follows from the composition of pullbacks and pushforwards, Lemma 1.1.15, and the commutative diagram

$$\begin{array}{ccc} Y_{\mathrm{proet}} & \xrightarrow{f_{\mathrm{proet}}} & X_{\mathrm{proet}} \\ \epsilon\downarrow & & \downarrow\epsilon \\ Y_{\mathrm{et}} & \xrightarrow{f_{\mathrm{et}}} & X_{\mathrm{et}} \end{array}$$

which comes from the naturality of ϵ.
Let $i : Y \to X$ be a closed immersion with quasi-compact open complement $j : U \to X$. Then for any sheaf G on Y_{et} we have

$$(\epsilon^* i_{\mathrm{et}*} G)|_U = j^*_{\mathrm{proet}} \epsilon^* i_{\mathrm{et}*} G = \epsilon^* j^*_{\mathrm{et}} i_{\mathrm{et}*} G = 0. \tag{4.3}$$

The axioms assure that $\mathrm{Sh}(Y_{\mathrm{proet}})$ is the complementary closed subtopos of $\mathrm{Sh}(U_{\mathrm{proet}})$. Equation (4.3) implies that $\epsilon^* i_{\mathrm{et}*} G \cong i_{\mathrm{proet}*} i^*_{\mathrm{proet}} \epsilon^* i_{\mathrm{et}*} G$. A computation leads to

$$i_{\mathrm{proet}*} i^*_{\mathrm{proet}} \epsilon^* i_{\mathrm{et}*} G = i_{\mathrm{proet}*} \epsilon^* i^*_{\mathrm{et}} i_{\mathrm{et}*} G \cong i_{\mathrm{proet}*} \epsilon^* G.$$

We used the equality $i^*_{\mathrm{et}} i_{\mathrm{et}*} G \cong G$ coming from Proposition 2.1.12. This proves the second claim.
For 3 let $V/U \in U_{\mathrm{et}}$. A section-wise calculation yields

$$f^*_{\mathrm{et}} \epsilon_* F(V/U) = \epsilon_* F(V) = F(\epsilon^{-1}(V)) = f^*_{\mathrm{proet}} F(\epsilon^{-1}(V)/\epsilon^{-1}(U)) = \epsilon_* f^*_{\mathrm{proet}} F(V/U).$$

which proves the first part of 3. The second part follows immediately from the adjointness properties. □

In order to gain a better understanding of constructible sheaves on the pro-étale site, we will make use of the short exact sequence (4.1) to obtain an equivalent definition of constructibility.

Proposition 4.3.12. A sheaf $F \in \mathrm{Sh}(X_{\mathrm{proet}})$ is constructible if and only if for every affine open $U \subset X$ there is a finite partition $U = \bigsqcup_{i=1}^n U_i$ in constructible locally closed subsets of U such that $F|_{U_i}$ is isomorphic to $\epsilon^* G_i$, where G_i is a locally constant étale sheaf of finite type on $U_{i\mathrm{et}}$.

Proof. ($\Rightarrow$) If F is constructible, it is of the form $\epsilon^* G$ for a constructible étale sheaf G. Then for any affine open $U \subset X$ there is a finite partition $U = \bigsqcup_{i=1}^n U_i$ in constructible locally closed subsets of U such that $G|_{U_i}$ is locally constant. Hence, $F|_{U_i} \cong (\epsilon^* G)|_{U_i} \cong \epsilon^*(G|_{U_i})$ is the pullback of a locally constant étale sheaf.

($\Leftarrow$)For the other implication we will first show that the canonical morphism

$$\epsilon^* \epsilon_* F \to F$$

is an isomorphism. As we can do this locally, we can assume that X is affine. Let $X = \bigsqcup_{i=1}^n U_i$ be the partition from the assumption. We proceed by induction on n. If

$n = 1$ then there is nothing to prove.
Assume $n > 1$. For every U_i there is a decomposition

$$U_i \overset{i}{\hookrightarrow} V_i \overset{j}{\hookrightarrow} X,$$

where i is a closed and j an open immersion. Again, it is enough to check the isomorphism on the V_i. So without loss of generality assume that U_i is a closed subscheme of X for some $i \in \{1, \ldots, n\}$. Let $i : Z := U_i \hookrightarrow X$ be the closed immersion with open complement $j : U \hookrightarrow X$. By induction hypothesis i^*F and j^*F are of the form $i^*F \cong \epsilon^*G_1$ and $j^*F \cong \epsilon^*G_2$ for étale sheaves G_1, G_2. In particular, Lemma 4.3.11 implies

$$j_!j^*F \cong \epsilon^* j_! G_1 \quad \text{and} \quad i_*i^*F \cong \epsilon^* i_* G_2.$$

Then Proposition 4.2.19 gives a short exact sequence

$$0 \to j_!j^*F \to F \to i_*i^*F \to 0,$$

where $j_!j^*F$ and i_*i^*F lie in the essential image of ϵ^*. Proposition 4.3.9 shows that F is in the essential image of ϵ^* and in particular $\epsilon^*\epsilon_*F \cong F$. It remains to show that $G := \epsilon_*\mathcal{F}$ is constructible. For any affine open $U \subset X$ there is a finite partition $U = \bigsqcup_{i=1}^n U_i$ in constructible locally closed subsets of U such that $F|_{U_i}$ is isomorphic to ϵ^*G_i, where G_i is a locally constant étale sheaf on $U_{i\text{et}}$. Especially, $\epsilon^*G|_{U_i} \cong \epsilon^*G_i$ and hence by the fully faithfulness of ϵ^* we have that $G|_{U_i} \cong G_i$ is locally constant. This proves the claim, as now G is by definition a constructible sheaf. □

Lemma 4.3.13. Let $I \subset R$ be an ideal and $j : U \hookrightarrow X$ a quasi-compact open immersion. Then for any sheaf F of R-modules on U_{proet} we have the formula

$$j_!IF \cong Ij_!F.$$

Proof. Denote by $i : Z \hookrightarrow X$ as the complement of U in X. An elementary computation shows $i^*Ij_!F = Ii^*j_!F = 0$. The short exact sequence from Proposition 4.2.19 implies

$$Ij_!F \cong j_!j^*Ij_!F \cong j_!Ij^*j_!F \cong j_!IF.$$

□

The Pro-étale Site for Open Immersions

Notation 4.3.14. Let U be an étale scheme over X. If the meaning is clear, we will sometimes denote the element $\epsilon^{-1}(U) \in X_{\text{proet}}$ by U to simplify notation.

Proposition 4.3.15 $(*)$**.** Let X be a scheme and $j : U \hookrightarrow X$ an open immersion. Assume that X_{proet} together with $\epsilon : X_{\text{proet}} \to X_{\text{et}}$ fulfill the Axioms 1 and 2. Then both axioms are also true for X_{proet}/U together with the induced map

$$\epsilon' : X_{\text{proet}}/U \longrightarrow X_{\text{et}}/U.$$

Proof. To show that X_{proet}/U has enough weakly contractible objects, it suffices to prove that any $V \to U$ is weakly contractible in X_{proet}/U if V is weakly contractible in X_{proet}. Let $\mathcal{F}_U \to \mathcal{G}_U$ be a surjective map of sheaves of sets in X_{proet}/U and $V/U \in X_{\text{proet}}/U$ an object such that V is weakly contractible in X_{proet}. As $j_{U!}$ preserves epimorphisms by Lemma 4.2.9, we get a surjection

$$j_{U!}\mathcal{F}_U(V) \longrightarrow j_{U!}\mathcal{G}_U(V).$$

Using the explicit formula for j_U^* in Lemma 1.1.26 one obtains that the map

$$j_U^* j_{U!}\mathcal{F}_U(V/U) = j_{U!}\mathcal{F}_U(V) \longrightarrow j_{U!}\mathcal{G}_U(V) = j_U^* j_{U!}\mathcal{G}_U(V/U).$$

is surjective. As $U \hookrightarrow X$ is a monomorphism, we can apply Lemma 4.2.10. Now, $j_U^* j_{U!} \cong \mathrm{id}$ which finally proves that $\mathcal{F}_U(V/U) \to \mathcal{G}_U(V/U)$ is surjective and hence that $V \to U$ is weakly contractible in X_{proet}/U.
The morphism of sites $\epsilon : X_{\text{proet}} \to X_{\text{et}}$ induces a morphism of sites

$$\epsilon' : X_{\text{proet}}/U \longrightarrow X_{\text{et}}/U.$$

We claim that for an abelian sheaf $\mathcal{F}_U$ on X_{et}/U we have

$$R^i\epsilon'_*\epsilon'^*\mathcal{F}_U = \begin{cases} \mathcal{F}_U & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$$

For an intermediate step, let $\mathcal{F}$ be an abelian sheaf on X_{proet} and choose an injective resolution $\mathcal{F} \to I^\bullet$. As j_U^* admits an exact left adjoint it follows from Lemma 4.1.1 that j_U^* preserves injectives. Even better, as j_U^* is exact it preserves injective resolutions. Hence $j_U^*\mathcal{F} \to j_U^* I^\bullet$ is an injective resolution of $j_U^*\mathcal{F}$. A direct computation shows that $\epsilon'_* j_U^* = j_{U,\text{et}}^*\epsilon_*$, which in turn implies

$$R^i\epsilon'_* j_U^*\mathcal{F} = H^i(\epsilon'_* j_U^* I^\bullet) = H^i(j_{U,\text{et}}^*\epsilon_* I^\bullet) = j_{U,\text{et}}^* H^i(\epsilon_* I^\bullet) = j_{U,\text{et}}^* R^i\epsilon_*\mathcal{F}.$$

With this result in mind, we can finally finish the proof, using the calculation

$$\begin{aligned} R^i\epsilon'_*\epsilon'^*\mathcal{F}_U &= R^i\epsilon'_*\epsilon'^* j_U^* j_{U!}\mathcal{F}_U \\ &= R^i\epsilon'_*(j_U \circ \epsilon')^*(j_{U!}\mathcal{F}_U) \\ &= R^i\epsilon'_*(\epsilon \circ j_U)^*(j_{U!}\mathcal{F}_U) \\ &= R^i\epsilon'_* j_U^*(\epsilon^* j_{U!}\mathcal{F}_U) \\ &= j_U^* R^i\epsilon_*\epsilon^*(j_{U!}\mathcal{F}_U). \end{aligned}$$

The claim follows as we have the respective result for $R^i\epsilon_*\epsilon^*$ and by Lemma 4.2.10 the identity $j_U^* j_{U!} \cong \mathrm{id}$. □

Remark 4.3.16. One should ask why we demanded Axiom 3 to work only for open immersions and not for all étale morphisms. The reason is exactly the proof of Proposition 4.3.15. We used the assumption that open immersions are monomorphisms in the category of schemes over X in order to use Lemma 4.2.10. In [7, Theorem 17.9.1], it is proven that étale monomorphisms are exactly the open immersions. Nevertheless, in both version of the pro-étale site presented in Section 4.7 the statement of the above proposition is true for any étale morphism $U \to X$.

Example 4.3.17. Let X be a scheme. Assume one has a site X_{proet} together with a morphism $\epsilon : X_{\text{proet}} \to X_{\text{et}}$ such that Axioms 1 and 2 are fulfilled. Let $E \subset \text{Sch}$ be the subcategory, which has open subschemes of X as objects and inclusions as morphisms. Proposition 4.3.15 shows that one can define a pro-étale enlargement for E by

$$U_{\text{proet}} := X_{\text{proet}}/U$$

for an open immersion $U \hookrightarrow X$.

4.4 Adic Pro-étale Sheaves

In this section we define adic sheaves for the pro-étale site. We will see that this definition is equivalent to the classical version introduced in Chapter 3 and derive important properties of the category of adic sheaves. For this section let $((-)_{\text{proet}}, \epsilon)$ be a pro-étale enlargement for $E = \text{Sch}$. Further let X be a scheme, Λ a ring and $I \subset \Lambda$ an ideal.

Definition 4.4.1. A sheaf of Λ-modules $\mathcal{F}$ is called *constructible* Λ*-sheaf* or I*-adic sheaf on* X_{proet} if $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of $\Lambda/I^n\Lambda$-modules for every $n \in \mathbb{N}$ and

$$\mathcal{F} \cong \lim_n \mathcal{F}/I^n\mathcal{F}.$$

If I is finitely generated, this is indeed equivalent to the classical definition in the following sense.

Theorem 4.4.2 $(*)$**.** Assume $I \subset \Lambda$ is finitely generated. Then there is a well-defined equivalence of categories

$$\phi : \left\{ \begin{array}{c} I\text{-adic sheaves on} \\ X_{\text{proet}} \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} I\text{-adic sheaves on} \\ X_{et} \end{array} \right\}$$

$$\mathcal{F} \longmapsto (\epsilon_*(\mathcal{F}/I^{n+1}\mathcal{F}))_{n\in\mathbb{N}}.$$

Moreover, the quasi-inverse for ϕ is given by

$$(G_n)_{n\in\mathbb{N}} \longmapsto \lim_n \epsilon^* G_n =: \psi((G_n)_{n\in\mathbb{N}}).$$

Proof. We will first show that ϕ is well-defined. We have to prove that for an I-adic pro-étale sheaf $\mathcal{F}$ the inverse system $\phi(\mathcal{F})$ is an I-adic étale sheaf. By definition $\mathcal{F}/I^{n+1}\mathcal{F}$ is constructible and therefore the pullback of an étale constructible sheaf G_n. In particular,

$$\epsilon_*(\mathcal{F}/I^{n+1}\mathcal{F}) = \epsilon_*\epsilon^* G_n = G_n$$

is constructible for all $n \in \mathbb{N}$. In order to prove that $I^{n+1}G_n = 0$, we apply the fully faithful functor ϵ^* and show $\epsilon^*(I^{n+1}G_n) = 0$ instead. Using Corollary 1.1.38 we compute

$$\epsilon^*(I^{n+1}G_n) = I^{n+1}\epsilon^* G_n = I^{n+1}(\mathcal{F}/I^{n+1}\mathcal{F}) = 0.$$

This proves $I^{n+1}G_n = 0$. We further have

$$\begin{aligned}\epsilon^*(G_{n+1}/I^{n+1}G_{n+1}) &= \epsilon^* G_{n+1}/\epsilon^*(I^{n+1}G_{n+1}) \\ &= \epsilon^* G_{n+1}/I^{n+1}(\epsilon^* G_{n+1}).\end{aligned}$$

The latter is isomorphic to $\epsilon^* G_n = \mathcal{F}/I^{n+1}\mathcal{F}$ as $\epsilon^* G_{n+1} = \mathcal{F}/I^{n+1}\mathcal{F}$. The fully faithfulness of ϵ^* implies $G_{n+1}/I^n G_{n+1} \cong G_n$ and proves that $\phi(\mathcal{F})$ is an I-adic étale sheaf.

Conversely, let $(F_n)_{n\in\mathbb{N}}$ be an I-adic étale sheaf. Define

$$\mathcal{F} := \lim_n \epsilon^* F_n.$$

Claim: There is a canonical isomorphism $\mathcal{F}/I^{n+1}\mathcal{F} \cong \epsilon^* F_n$.
The projection to the n'th component yields a map $\mathcal{F} \to \epsilon^* F_n$. As $I^{n+1}\epsilon^* F_n = \epsilon^* I^{n+1} F_n$ vanishes, we get a map $\mathcal{F}/I^{n+1}\mathcal{F} \to \epsilon^* F_n$. It remains to show that this is an isomorphism. By Proposition 4.2.28 it is enough to check this on all weakly contractible objects of X_{proet}. That is, $\Gamma(U, \mathcal{F}/I^{n+1}\mathcal{F}) \cong \Gamma(U, \epsilon^* F_n)$ for all weakly contractible $U \in X_{\text{proet}}$. As an intermediate step, we want to show that $(\epsilon^* F_n(U))_n$ is an I-adic system of Λ-modules. Using the exactness of $\Gamma(U, -)$ and Lemma 4.2.29, we compute

$$\begin{aligned}(\epsilon^* F_{n+1}(U))/I^{n+1}(\epsilon^* F_{n+1}(U)) &\cong (\epsilon^* F_{n+1}(U))/(\epsilon^* I^{n+1} F_{n+1}(U)) \\ &\cong \left(\epsilon^* F_{n+1}/\epsilon^* I^{n+1} F_{n+1}\right)(U).\end{aligned}$$

As ϵ^* is exact, we see

$$\epsilon^* F_{n+1}/\epsilon^* I^{n+1} F_{n+1} \cong \epsilon^*(F_{n+1}/I^{n+1}F_{n+1}) \cong \epsilon^* F_n.$$

Summerizing, one can say $(\epsilon^* F_{n+1}(U))/I^{n+1}(\epsilon^* F_{n+1}(U)) \cong \epsilon^* F_n(U)$, which proves that $(\epsilon^* F_n(U))_n$ is an I-adic system of modules. We can apply Proposition 1.3.7 to the system $(\epsilon^* F_n(U))_{n\in\mathbb{N}}$ to see that $\mathcal{F}(U)$ is an I-adically complete Λ-module and that

$$\mathcal{F}(U)/I^{n+1}\mathcal{F}(U) \cong \epsilon^* F_n(U).$$

Finally, $\epsilon^* F_n(U) \cong (\mathcal{F}/I^{n+1}\mathcal{F})(U)$, which completes the proof of the claim, i.e. we have shown $\mathcal{F}/I^{n+1}\mathcal{F} \cong \epsilon^* F_n$.

The remaining part just consists of formalities. To ensure completeness, we will provide all the details. The claim shows that $\mathcal{F}$ is indeed an I-adic sheaf on X_{proet}, i.e. the functor $(F_n)_{n\in\mathbb{N}} \mapsto \lim_n \epsilon^* F_n := \psi((F_n)_{n\in\mathbb{N}})$ is well-defined. Further,

$$\phi \circ \psi((F_n)_{n\in\mathbb{N}}) = (\epsilon_*\epsilon^* F_n)_{n\in\mathbb{N}} \cong (F_n)_{n\in\mathbb{N}} \quad \text{and}$$
$$\psi \circ \phi(\mathcal{F}) = \lim_n \epsilon^*\epsilon_*\mathcal{F}/I^{n+1}\mathcal{F} = \lim_n \mathcal{F}/I^{n+1}\mathcal{F} \cong \mathcal{F}.$$

Here, $\epsilon^*\epsilon_*\mathcal{F}/I^{n+1}\mathcal{F} \cong \mathcal{F}/I^{n+1}\mathcal{F}$ as $\mathcal{F}/I^{n+1}\mathcal{F}$ is in the essential image of ϵ^*. This proves that ψ and ϕ are quasi-inverse functors and in particular that ϕ is an equivalence of categories. □

Remark 4.4.3. In the proof of Theorem 4.4.2 we did not speak of constructibility in the sense of its definition. More precisely, we could replace the assumptions of the foregoing theorem by the following data. Let $\mathcal{S} \subset \mathrm{Mod}_\Lambda(X_{\text{et}})$ be a strictly full subcategory. We call $F \in \mathrm{Mod}_\Lambda(X_{\text{et}})$ an adic $\mathcal{S}$-sheaf if F is I-adically complete and $F/I^n F$ is isomorphic to $\epsilon^* G_n$ for a sheaf $G_n \in \mathcal{S}$. An inverse system $(F_n)_{n\in\mathbb{N}}$ of sheaves on X_{et} is called I-adic $\mathcal{S}$-sheaf, if $F_n \cong F_{n+1}/I^{n+1}F_{n+1}$ and $F_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. Then the proof of Theorem 4.4.2 shows that

$$\phi : \left\{ \begin{array}{c} I\text{-adic } \mathcal{S}\text{-sheaves on} \\ X_{\text{proet}} \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} I\text{-adic } \mathcal{S}\text{-sheaves on} \\ X_{\text{et}} \end{array} \right\}$$

is an equivalence of categories. Later statements cannot be generalized in this way, as we need the finiteness conditions of constructibility to apply the theory developed in Chapter 3.

Proposition 4.4.4. Extension by zero along a quasi-compact open subset preserves I-adic pro-étale sheaves. That is, if $j : U \hookrightarrow X$ is a quasi-compact open immersion and F an I-adic sheaf on U_{proet}, then $j_! F$ is an I-adic sheaf on X_{proet}.

Proof. We will first show that $j_! F$ is I-adically complete. Consider the I-adically complete sheaf

$$G := \lim_n j_! F / I^n j_! F$$

and compute with Axiom 5 for the complement $i : Z := X \setminus U \hookrightarrow X$ that

$$G|_Z = i^* \lim_n j_! F / I^n j_! F = \lim_n i^* \left(j_! F / I^n j_! F \right) = 0.$$

The short exact sequence from Proposition 4.2.19 then implies that $G = j_! H$ for a sheaf $H \in \mathrm{Sh}(U_{\text{proet}})$. We can compute $H \cong j^* G \cong \lim_n j^* j_! F / j^* I^n j_! F \cong \lim_n F / I^n F$, which is the I-adic completion of F in $\mathrm{Sh}(U_{\text{proet}})$. As F is I-adically complete, this implies

$H \cong F$, which proves $j_! F \cong G$. This implies the completeness of F.
For the constructibility of $j_! F$ we can use the methods from Section 4.3 to show

$$j_! F / I^n j_! F \cong j_! \left(F / I^n F \right).$$

By assumption we can write $F/I^n F$ as $\epsilon^* G_n$ for a constructible sheaf G_n on U_{et}. We then have by Lemma 4.3.11

$$j_! F / I^n j_! F \cong j_! \epsilon^* G_n = \epsilon^* j_! G_n.$$

Hence, $j_! F / I^n j_! F$ is the pullback of the constructible étale sheaf $j_! G_n$. $\square$

Proposition 4.4.5. Assume $i : Z \hookrightarrow X$ is a closed immersion with quasi-compact complement $j : U \hookrightarrow X$. Further, let $\mathcal{F}$ be an I-adic sheaf on Z_{proet}. Then $i_* \mathcal{F}$ is I-adic.

Proof. Note that i_* has a left adjoint i^* and a right adjoint $i^!$, see Proposition 4.2.20 and Proposition 1.1.14. Therefore, i_* commutes with small limits and colimits. Moreover, Corollary 1.1.38 implies that for any ideal $J \subset R$ the equation

$$j^* J i_* \mathcal{F} \cong J j^* i_* \mathcal{F} = 0$$

holds. As $\mathrm{Sh}(Z_{\mathrm{proet}})$ is the complementary closed subtopos of $\mathrm{Sh}(U_{\mathrm{proet}})$, we have

$$J i_* \mathcal{F} \cong i_* i^* J i_* \mathcal{F} \cong i_* J i^* i_* \mathcal{F} \cong i_* J \mathcal{F}.$$

Now we can show completeness of $i_* \mathcal{F}$. We compute

$$\lim_n i_* \mathcal{F} / I^n i_* \mathcal{F} \cong \lim_n i_* \left(\mathcal{F} / I^n \mathcal{F} \right) \cong i_* \lim_n \left(\mathcal{F} / I^n \mathcal{F} \right) \cong i_* \mathcal{F}.$$

The constructibility of $\mathcal{F}/I^n \mathcal{F}$ is immediate using Lemma 4.3.11 and the above calculations. $\square$

The Pro-étale Site for Noetherian Rings and Noetherian Schemes

In the following we will discuss properties of I-adic sheaves if X is a noetherian scheme and R a noetherian ring. Remarkably, this assumption is sufficient to attain that the adic pro-étale sheaves form an abelian subcategory of $\mathrm{Mod}_R(X_{\mathrm{proet}})$. Recall that in the classical case we had to pass to the AR-category to obtain that property. In the pro-étale world there is no need for such formality. Note that this section strongly relies on the Artin-Rees lemma which is why we demand R to be noetherian.

Remark 4.4.6. As ϵ^* is an exact functor, the functor

$$\epsilon^* : \{\text{Inverse systems of étale sheaves}\} \to \{\text{Inverse systems of pro-étale sheaves}\}$$
$$(F_n)_{n \in \mathbb{N}} \mapsto (\epsilon^* F_n)_{n \in \mathbb{N}}$$

is well-defined and exact. Moreover, it maps null-systems to null-systems, which is why we can apply the universal property of the AR-category in Proposition 1.2.3 to get an exact functor

$$\epsilon^* : \mathrm{AR}(\mathrm{Ab}(X_{\mathrm{et}})) \to \mathrm{AR}(\mathrm{Ab}(X_{\mathrm{proet}})).$$

For the following we assume that all considered schemes and rings are noetherian. Recall that in this case the AR I-adic sheaves on X_{et} form an abelian subcategory of $\text{AR}(\text{Mod}_R(X_{\text{proet}}))$ by Proposition 3.4.1. We can formulate the following theorem.

Theorem 4.4.7 (*)**.** Let X be a noetherian scheme and Λ a noetherian ring. Then we obtain an exact functor

$$\begin{aligned} \lim_{\text{AR}}\epsilon^* : \{\text{AR } I\text{-adic sheaves on } X_{\text{et}}\} &\to \text{Mod}_R(X_{\text{proet}}) \\ (F_n)_{n\in\mathbb{N}} &\mapsto \lim_n \epsilon^* F_n. \end{aligned}$$

Proof. Let

$$0 \to (F_n)_n \to (G_n)_n \to (H_n)_n \to 0$$

be a short exact sequence in the AR-Category of sheaves on X_{et}, such that all systems involved are I-adic. Then by the previous remark we get a short exact sequence

$$0 \to (\epsilon^* F_n)_n \to (\epsilon^* G_n)_n \to (\epsilon^* H_n)_n \to 0$$

in the AR-Category of systems of sheaves on X_{proet}. Applying Rlim_{AR} we obtain a long exact sequence

$$0 \to \lim_{\text{AR}}(\epsilon^* F_n)_n \to \lim_{\text{AR}}(\epsilon^* G_n)_n \to \lim_{\text{AR}}(\epsilon^* H_n)_n \to \text{R}^1\lim_{\text{AR}}(\epsilon^* F_n)_n \to \cdots.$$

But $(\epsilon^* F_n)_n$ is an inverse system with surjective transition maps, as it is I-adic. Hence, by Corollary 4.1.7 and Proposition 4.2.34 we have

$$\text{R}^1\lim_{\text{AR}}(\epsilon^* F_n)_n = \text{R}^1\lim(\epsilon^* F_n)_n = 0.$$

This proves that $\lim_{\text{AR}}\epsilon^*$ is exact. □

Corollary 4.4.8 (*)**.** Let X be a noetherian scheme and R a noetherian ring. Then the category of I-adic sheaves on X_{proet} is an abelian subcategory of $\text{Mod}_R(X_{\text{proet}})$.

Proof. By Theorem 4.4.2 and Theorem 4.4.7 we see that the category of I-adic sheaves on X_{proet} is exactly the essential image of the fully faithful exact functor

$$\lim_{\text{AR}}\epsilon^* : \{\text{AR } I\text{-adic sheaves on } X_{\text{et}}\} \to \text{Mod}_R(X_{\text{proet}}).$$

This proves the claim. □

Proposition 4.4.9 (*)**.** Let X be a noetherian scheme and R a noetherian ring. Then the category of I-adic sheaves on X_{proet} is a noetherian category, i.e. if $\mathcal{F}$ is an I-adic sheaf, then every increasing chain

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}$$

of I-adic subsheaves of $\mathcal{F}$ gets eventually constant.

Proof. This is a direct consequence of Proposition 3.4.3. □

Proposition 4.4.10 (∗)**.** Let X be a noetherian scheme and R a noetherian ring. The category of I-adic sheaves on X_{proet} is stable under extensions. That means the following. If there is a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

where $\mathcal{F}$ and $\mathcal{H}$ are I-adic, then $\mathcal{G}$ is an I-adic sheaf.

Proof. First, we have to prove the completeness of $\mathcal{G}$. By the strategy of the proof of Theorem 4.4.2, it is equivalent to check the completeness for $\mathcal{G}(U)$, where U runs over all weakly contractible objects of X_{proet}. As I is finitely generated, we can write $I = (a_1, \dots, a_n)$. By Lemma 1.3.11 it suffices to prove that $\mathcal{G}$ is a_i-adic complete for all $i \in \{1, \dots, n\}$ and therefore we assume that $I = (a)$ is a principle ideal. Consider for arbitrary $n \in \mathbb{N}$ the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\cdot a^n"] & \mathcal{G} \arrow[r] \arrow[d, "\cdot a^n"] & \mathcal{H} \arrow[r] \arrow[d, "\cdot a^n"] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. We call this diagram D_n. There is a morphism of commutative diagrams $D_{n+1} \to D_n$ given by multiplication by a in the top row and taking the identity in the lower row. As the snake lemma also includes a functoriality result, we obtain from $D_{n+1} \to D_n$ a diagram with exact rows

$$\begin{tikzcd}
\mathcal{H}[a^{n+1}] \arrow[r] \arrow[d, "\cdot a"] & \mathcal{F}/a^{n+1}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{G}/a^{n+1}\mathcal{G} \arrow[r] \arrow[d] & \mathcal{H}/a^{n+1}\mathcal{H} \arrow[r] \arrow[d] & 0 \\
\mathcal{H}[a^n] \arrow[r] & \mathcal{F}/a^n\mathcal{F} \arrow[r] & \mathcal{G}/a^n\mathcal{G} \arrow[r] & \mathcal{H}/a^n\mathcal{H} \arrow[r] & 0.
\end{tikzcd} \tag{4.4}$$

Here, $\mathcal{H}[a^{n+1}]$ is the kernel of $\mathcal{H} \xrightarrow{a^n} \mathcal{H}$ by definition. By Corollary 4.4.8, $\mathcal{H}[a^{n+1}]$ is an I-adic sheaf. Furthermore, Proposition 4.4.9 implies that the sequence

$$\mathcal{H}[a] \subset \mathcal{H}[a^2] \subset \cdots \subset \mathcal{H}[a^n] \subset \cdots \subset \mathcal{H}$$

gets eventually constant. This means that there is an $N \in \mathbb{N}$ with $\mathcal{H}[a^m] = \mathcal{H}[a^N]$ for all $m \geq N$. In particular, the inverse system $(\mathcal{H}[a^n], \cdot a)_{n \in \mathbb{N}}$, which is on the left of Eq. (4.4), is AR-zero[6]. We conclude that

$$0 \to (\mathcal{F}/a^n\mathcal{F})_n \to (\mathcal{G}/a^n\mathcal{G})_n \to (\mathcal{H}/a^n\mathcal{H})_n \to 0$$

[6] The shift by N is the zero morphism.

is AR-exact. As $(\mathcal{F}/a^n\mathcal{F})_n$ is a system with surjective transition maps, Corollary 4.1.7 and Proposition 4.2.34 imply that $\mathrm{R}^1\lim_{\mathrm{AR}}\mathcal{F}/a^n\mathcal{F} \cong \mathrm{R}^1\lim_n\mathcal{F}/a^n\mathcal{F} = 0$, which in turn implies that

$$0 \to \lim_n\mathcal{F}/a^n\mathcal{F} \to \lim_n\mathcal{G}/a^n\mathcal{G} \to \lim_n\mathcal{H}/a^n\mathcal{H} \to 0$$

is exact. The 5-lemma applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\cong"'] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \lim_n\mathcal{F}/a^n\mathcal{F} \arrow[r] & \lim_n\mathcal{G}/a^n\mathcal{G} \arrow[r] & \lim_n\mathcal{H}/a^n\mathcal{H} \arrow[r] & 0
\end{tikzcd}$$

finishes the proof of completeness.

It remains to show that $\mathcal{G}/I^n\mathcal{G}$ is a constructible sheaf on X_{proet}. We can write $I^n = (a_1, \ldots, a_l)$ and proceed by induction on l.
If $I^n = (a)$ for an $a \in R$, define $\mathcal{F}_n := \mathcal{F}/a\mathcal{F}$, $\mathcal{G}_n := \mathcal{G}/a\mathcal{G}$ and $\mathcal{H}_n := \mathcal{H}/a\mathcal{H}$ and obtain by the snake lemma an exact sequence

$$\mathcal{H}[a] \xrightarrow{\delta} \mathcal{F}_n \to \mathcal{G}_n \to \mathcal{H}_n \to 0$$

as above. Note that $\mathcal{F}_n$ and $\mathcal{G}_n$ are constructible sheaves on X_{proet}. Further, the sheaf $\mathcal{H}[a] = \ker(\mathcal{H} \xrightarrow{\cdot a} \mathcal{H})$ is an I-adic sheaf by Corollary 4.4.8. As $a\mathcal{H}[a] = 0$, we have $\mathcal{H}[a]/a\mathcal{H}[a] = \mathcal{H}[a]$ and therefore $\mathcal{H}[a]$ is also the pullback of a constructible étale sheaf. By Corollary 4.3.10 we conclude that $\mathcal{G}_n$ is an I-adic sheaf on X_{proet}.
If $l > 1$ set $a := a_1$ and define $\mathcal{F}_n := \mathcal{F}/a\mathcal{F}$, $\mathcal{G}_n := \mathcal{G}/a\mathcal{G}$ and $\mathcal{H}_n := \mathcal{H}/a\mathcal{H}$. We get an exact sequence

$$\mathcal{H}[a] \xrightarrow{\delta} \mathcal{F}_n \to \mathcal{G}_n \to \mathcal{H}_n \to 0.$$

Corollary 4.4.8 implies that $\mathcal{H}[a] = \ker(\mathcal{H} \xrightarrow{\cdot a} \mathcal{H})$, $\mathcal{F}_n = \mathrm{coker}(\mathcal{F} \xrightarrow{\cdot a} \mathcal{F})$ and $\mathcal{H}_n$ are I-adic sheaves. All involved sheaves have a canonical $\Lambda' := \Lambda/(a)$-module structure and $\mathcal{F}_n$, $\mathcal{H}_n$ and $\mathcal{H}[a]$ are I'-adic sheaves of R'-modules with respect to the ideal $I' = I/(a) \subset R'$. In particular, $\mathrm{im}(\delta)$ is an I'-adic sheaf. Using this, we get a short exact sequence

$$0 \to (\mathcal{F}/a\mathcal{F})/\mathrm{im}(\delta) \to \mathcal{G}/a\mathcal{G} \to \mathcal{H}/a\mathcal{H} \to 0,$$

where both, the left and the right sheaf are I'-adic sheaves. As I'^n is generated by $l-1$ elements, the induction hypothesis implies that $\mathcal{G}_n = \mathcal{G}/a\mathcal{G}$ is an I'-adic sheaf. In particular,

$$\mathcal{G}_n/I'^n\mathcal{G}_n \cong \mathcal{G}/I^n\mathcal{G}$$

is the pullback of a constructible sheaf on X_{et}. $\square$

Corollary 4.4.11 $(*)$**.** The category of I-adic sheaves on X_{proet} forms a weak Serre subcategory of $\text{Mod}_R(X_{\text{proet}})$.

Proof. This is an immediate consequence of Corollary 4.4.8 and Proposition 4.4.10. □

Corollary 4.4.12. Let $\mathcal{F}$ be a sheaf of R-modules on X_{proet} and $j : U \hookrightarrow X$ an open immersion immersion and consider its closed complement $i : Z \hookrightarrow X$. Assume $\mathcal{F}|_U$ and $\mathcal{F}|_Z$ are I-adic, then $\mathcal{F}$ is I-adic.

Proof. Proposition 4.2.19 gives a short exact sequence

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0.$$

By assumption, $j^*\mathcal{F}$ and $i^*\mathcal{F}$ are I-adic. This, in turn, implies that $j_! j^*\mathcal{F}$ and $i_* i^*\mathcal{F}$ are I-adic, see Proposition 4.4.4 and Proposition 4.4.5. We conclude using Proposition 4.4.10. □

Corollary 4.4.13. Let $\mathcal{F}$ be a sheaf of R-modules on X_{proet}. If there is a finite stratification[7] $X = \bigsqcup_{i=1}^n X_i$ such that $\mathcal{F}|_{X_i}$ is an I-adic sheaf, then $\mathcal{F}$ is I-adic.

Proof. First assume X is irreducible. We prove the claim by induction on the length of the partition. For $n = 1$ the assertion is clear.
Assume $n > 1$. Analogously to Corollary 3.3.8, one of the stata X_i is open in X. Let $j : U \hookrightarrow X$ be the associated open immersion and $i : Z \hookrightarrow X$ its closed complement. The induction hypothesis implies that $i^*\mathcal{F}$ and $j^*\mathcal{F}$ are I-adic. Now the claim follows from Corollary 4.4.12.

Now we come to the general case. Let $X_1, \ldots, X_m$ be the irreducible components of X. We use induction on m. If $m = 1$, X is irreducible. We already proved this case. For $m > 1$ define

$$U := X_1 \setminus \bigcup_{i=2}^{m} X_i.$$

Now, U is an open and irreducible subset of X. Hence, by the above $\mathcal{F}|_U$ is I-adic. The induction hypothesis applied to $Z := X \setminus U$ yields that $\mathcal{F}|_Z$ is I-adic. We conclude using Corollary 4.4.12. □

Definition 4.4.14. Let $\mathcal{F}$ be an I-adic sheaf. Then $\mathcal{F}$ is called *locally constant* or *lisse*[8] if each $\mathcal{F}/I^n\mathcal{F}$ is a locally constant sheaf of Λ-modules.

Corollary 4.4.15. Let $\mathcal{F}$ be an I-adic sheaf on the pro-étale site X_{proet}. Then there exists a finite partition $X = \bigsqcup_{i=1}^n X_i$ such that $F|_{X_i}$ is lisse.

[7] With our convention, a stratification is a partition consisting of constructible locally closed subsets.
[8] In this context we will usually prefer the term lisse to avoid confusion with the term locally constant sheaf of Λ-modules.

Proof. This is a direct consequence of Proposition 3.3.7. □

Corollary 4.4.16. Let $\mathcal{F}$ be an I-adic sheaf on the pro-étale site X_{proet}. Then there exists an open dense subset $U \subset X$ such that $F|_U$ is lisse.

Proof. This is the corresponding statement to Corollary 3.3.8. □

4.5 Cohomology of Adic Pro-étale Sheaves

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $h : \mathcal{A} \to \mathcal{B}$ a left exact functor. Consider the induced left exact functor

$$h^{\mathbb{N}} : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{B}^{\mathbb{N}}.$$

Assume inverse limits exist in $\mathcal{B}$, then there is a well-defined left exact functor

$$\lim_n : \mathcal{B}^{\mathbb{N}} \longrightarrow \mathcal{B}$$

sending a system to its inverse limit. We denote the composite $\lim_n \circ h^{\mathbb{N}}$ as $\lim_n h$. In [10] one can find the following definition.

Definition 4.5.1 (continuous étale cohomology)**.** Let X be a scheme and $(F_n, d_n)_{n\in\mathbb{N}}$ an inverse system of abelian sheaves on X_{et}. Then define the *continuous étale cohomology groups* as

$$H^i_{\text{cont}}(X_{\text{et}}, (F_n, d_n)_n) := R^i(\lim_n \Gamma)\left((F_n, d_n)_n\right)$$

Lemma 4.5.2. Let $\mathcal{A}$ be a Grothendiecke abelian category and $\mathcal{B}$ an abelian category. Assume that countable products are exact in $\mathcal{B}$ and that $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, which commutes with countable products. Then $\mathrm{R}F$ commutes with derived limits.

Proof. [15, Tag 08U1] □

Example 4.5.3. Lemma 4.5.2 can be applied to the sections functor

$$\Gamma(U, -) : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}.$$

Therefore, for any inverse system $(F_n, d_n)_{n\in\mathbb{N}}$ of abelian sheaves, we have

$$\mathrm{R}\lim_n \mathrm{R}\Gamma(U, F_n) = \mathrm{R}\Gamma(U, \mathrm{R}\lim_n F_n).$$

Lemma 4.5.4 (composition of derived functors)**.** Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ abelian categories and assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Assume $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are two left exact functors and F sends injective objects to G-right acyclic objects. Then the canonical map

$$\mathrm{R}(G \circ F) \to \mathrm{R}(G) \circ \mathrm{R}(F)$$

is an isomorphism of functors $D^+(\mathcal{A}) \to D^+(\mathcal{C})$.

Proof. [15, Tag 015M] ☐

Theorem 4.5.5 $(*)$**.** Let X be a scheme. Consider an inverse system of sheaves $(F_n)_{n\in\mathbb{N}}$ on the étale site X_{et} with surjective transition maps[9]. Then we have

$$H^i_{\text{cont}}(X_{\text{et}},(F_n)_{n\in\mathbb{N}}) \cong H^i(X_{\text{proet}},\lim_n \epsilon^* F_n)$$

Proof. Composition of derived functors implies $\mathrm{R}\left(\Gamma(X_{\text{et}},-)\circ\lim_n\right) = \mathrm{R}\Gamma(X_{\text{et}},-)\circ\mathrm{Rlim}_n$. We compute

$$\begin{aligned}
\mathrm{R}\Gamma\left(X_{\text{et}},\mathrm{Rlim}_n(F_n)_{n\in\mathbb{N}}\right) &\cong \mathrm{Rlim}_n(\mathrm{R}\Gamma(X_{\text{et}},F_n)) \\
&\cong \mathrm{Rlim}_n(\mathrm{R}\Gamma(X_{\text{proet}},\epsilon^* F_n)) \\
&\cong \mathrm{R}\Gamma\left(X_{\text{proet}},\mathrm{Rlim}_n(\epsilon^* F_n)_{n\in\mathbb{N}}\right) \\
&\cong \mathrm{R}\Gamma\left(X_{\text{proet}},\lim_n(\epsilon^* F_n)\right)
\end{aligned}$$

The first and third equation use the commutation of Rlim and RΓ from Example 4.5.3. The second equation is Proposition 4.3.6 and the last isomorphism comes from Proposition 4.2.34. ☐

Definition 4.5.6. Let $(F_n)_{n\in\mathbb{N}}$ be an I-adic sheaf. Define the *I-adic cohomology group of* $(F_n)_{n\in\mathbb{N}}$ as

$$H^i(X_{\text{et}},(F_n)_{n\in\mathbb{N}}) := \lim_n H^i(X_{\text{et}},F_n).$$

We have the following connection to continuous étale cohomology.

Proposition 4.5.7. Let $(F_n,d_n)_{n\in\mathbb{N}}$ be an inverse system of abelian sheaves on X_{et}. Then there is a canonical short exact sequence

$$0\to \lim^1_n H^{i-1}(X,F_n)\to H^i_{\text{cont}}(X,(F_n)_{n\in\mathbb{N}})\to H^i(X_{\text{et}},(F_n)_{n\in\mathbb{N}})\to 0.$$

Proof. [10, Equation (3.1)] ☐

Proposition 4.5.8 $(*)$**.** If the system $\left(H^{i-1}(X_{\text{et}},F_n)\right)_n$ fulfills the Mittag Leffler condition, then

$$H^i(X_{\text{proet}},\lim_n \epsilon^* F_n)\cong H^i_{\text{cont}}(X_{\text{et}},(F_n)_{n\in\mathbb{N}})\cong H^i(X_{\text{et}},(F_n)_{n\in\mathbb{N}})$$

Proof. The first isomorphism comes from Theorem 4.5.5. The second isomorphism comes from the short exact sequence in Proposition 4.5.7 and from the vanishing of $\lim^1_n H^{i-1}(X_{\text{et}},F_n)$ by Lemma 4.1.4. ☐

[9] E.g. an I-adic sheaf.

4.6 Constructible Complexes vs Adic Pro-étale Sheaves

In their paper [2] Bhatt and Scholze introduced a derived category $\mathrm{D}_{\mathrm{cons}}(X_{\mathrm{proet}}, R) \subset \mathrm{D}(X_{\mathrm{proet}}, R)$, whose objects are called *constructible complexes*. In this subsection we want to draw a connection to our definition of I-adic sheaves on X_{proet} in some special cases. At the end, all statements should work for $R = \mathbb{Z}_\ell$ and $I = (\ell)$. This section uses the notion of derived categories without further explanation. We will sketch many preliminary statements and instead of detailed proofs refer to the literature. However, the goal of this chapter is achieved in Theorem 4.6.21. Let X be a noetherian scheme and R a noetherian ring. We write $\mathrm{D}(X_{\mathrm{proet}}, R)$ for the derived category of $\mathrm{Mod}_R(X_{\mathrm{proet}})$.

Notation 4.6.1. We use the following notation. If $\mathcal{G} \in \mathrm{Mod}_R(X_{\mathrm{proet}})$ and $i \in \mathbb{Z}$ we write $\mathcal{G}[i]$ for the complex

$$\cdots \to 0 \to \mathcal{G} \to 0 \to \cdots,$$

where $\mathcal{G}$ sits in degree i. If it is clear from the context, we will also write $\mathcal{G}$ for $\mathcal{G}[0]$.

Definition 4.6.2 (derived complete)**.** Let $K \in \mathrm{D}(X_{\mathrm{proet}}, R)$. We say that K is *derived I-complete* if for all $U \in X_{\mathrm{proet}}$ and all $x \in I(U)$ the derived limit

$$T(K, x) := \mathrm{Rlim}(\cdots \xrightarrow{\cdot x} K \xrightarrow{\cdot x} K \xrightarrow{\cdot x} K) \in \mathrm{D}(X_{\mathrm{proet}}, R)$$

vanishes. We say that $\mathcal{F} \in \mathrm{Mod}_R(X_{\mathrm{proet}})$ is *derived complete* if $\mathcal{F}[0]$ is derived complete. Although, this seems to be the accepted definition we will mainly use the following lemma to deal with derived complete complexes.

Lemma 4.6.3. Assume R is a noetherian ring. Then a complex $K \in \mathrm{D}(X_{\mathrm{proet}}, R)$ is derived I-complete if and only if the canonical maps induce an isomorphism

$$K \cong \mathrm{Rlim}(K \otimes^{\mathrm{L}}_R R/I^n).$$

Proof. [2, Proposition 3.5.1] □

Lemma 4.6.4. A sheaf $\mathcal{F} \in \mathrm{Mod}_R(X_{\mathrm{proet}})$ is complete if and only if it is derived complete and $\bigcap_{n \in \mathbb{N}} I^n \mathcal{F} = 0$.

Proof. [15, Tag 099Q] □

First, we have to introduce constructible complexes on the étale site.

Definition 4.6.5. A complex $K \in \mathrm{D}(X_{\mathrm{et}}, R)$ is called *constructible* if there is a finite stratification $X = \bigsqcup_{i=1}^n X_i$ such that $K|_{X_i}$ is locally constant with perfect values on X_{et}. That is, $K|_{X_i}$ is locally quasi-isomorphic to a complex which comes from a bounded complex $L \in \mathrm{D}(R)$ of finite projective R-modules, see [2, Definition 6.3.1]. Also see the definition of a perfect complex in [15, Tag 0657].

In the paper of B. Bhatt and P. Scholze [2] one can find the following definition.

Definition 4.6.6. A complex $K \in \mathrm{D}(X_{\mathrm{proet}}, R)$ is called *constructible* if

- K is derived complete and
- The complex $K \otimes^{\mathrm{L}}_{R} R/I$ is the pullback of a constructible R/I complex on X_{et}.

We donote $\mathrm{D}_{\mathrm{cons}}(X_{\mathrm{proet}}, R) \subset \mathrm{D}(X_{\mathrm{proet}}, R)$ as the full subcategory of constructible complexes.

However, sometimes it is more useful to use the following equivalent definition from the Stacks Project [15, Tag 09C1].

Definition 4.6.7. A complex $K \in \mathrm{D}(X_{\mathrm{proet}}, R)$ is called *constructible* if

- K is derived complete and
- The complex $K \otimes^{\mathrm{L}}_{R} R/I$ has constructible cohomology sheaves and finite tor dimension.

Lemma 4.6.8. Definition 4.6.6 and Definition 4.6.7 are equivalent.

Proof. It is clear that Definition 4.6.6 implies 4.6.7. For the converse assume $K \otimes^{\mathrm{L}}_{R} R/I$ has constructible cohomology sheaves and finite tor dimension. In this case [15, Tag 09C2] implies that there is an isomorphism

$$K \otimes^{\mathrm{L}}_{R} R/I \cong \epsilon^* L$$

for a complex $L \in D(X_{\mathrm{et}}, R)$ with finite tor dimension and constructible cohomology sheaves. The claim then follows from [15, Tag 03TT]. $\square$

Proposition 4.6.9. Let $K \in \mathrm{D}^{-}(X_{\mathrm{proet}}, \Lambda)$ such that $K_1 := K \otimes^{\mathrm{L}}_{\Lambda} \Lambda/I$ has constructible cohomology sheaves. Then each $K_n := K \otimes^{\mathrm{L}}_{\Lambda} \Lambda/I^n$ has constructible cohomology sheaves.

Proof. We use induction on n to prove this claim. Let $n \geq 1$ and consider the distinguished triangle

$$K \otimes^{\mathrm{L}}_{\Lambda} I^n/I^{n+1} \to K_{n+1} \to K_n \to K \otimes^{\mathrm{L}}_{\Lambda} I^n/I^{n+1}[1] \tag{4.5}$$

which comes from applying $- \otimes^{\mathrm{L}}_{\Lambda} K$ to the canonial exact sequence

$$0 \to I^n/I^{n+1} \to \Lambda/I^{n+1} \to \Lambda/I^n \to 0.$$

Computations with the derived tensor product yields $K \otimes^{\mathrm{L}}_{\Lambda} I^n/I^{n+1} \cong K_1 \otimes^{\mathrm{L}}_{\Lambda} I^n/I^{n+1}$. In particular, $K_1 \otimes^{\mathrm{L}}_{\Lambda} I^n/I^{n+1}$ is a sequence with constructible cohomology sheaves, for instance see [15, Tag 0961]. Consider the long exact sequence of cohomology sheaves associated to (4.5) and finish the proof using that the constructible sheaves form a weak Serre subcategory of $\mathrm{Mod}_{\Lambda}(X_{\mathrm{proet}})$ (see Corollary 4.3.10). $\square$

Corollary 4.6.10. Let $\mathcal{F}$ be a sheaf on X_{proet} such that $\mathcal{F}[0] \otimes^{\mathrm{L}}_{\Lambda} \Lambda/I$ has constructible cohomology sheaves, then $\mathcal{F}/I^n\mathcal{F}$ is constructible for all $n \in \mathbb{N}$.

Proof. This follows directly from Proposition 4.6.9, as $\mathcal{F}/I^n\mathcal{F} = H^0(\mathcal{F}[0] \otimes^{\mathrm{L}}_{\Lambda} \Lambda/I^n)$. □

Definition 4.6.11. A *regular local ring* is a local noetherian ring R, such that for the maximal ideal $m \subset R$ the equality

$$\dim_{R/m}(m/m^2) = \dim(R)$$

holds. Here, $\dim(R)$ is the Krull dimension of R.

Definition 4.6.12. A *regular sequence on* R is a n-tupel $(x_1, \dots, x_n)$ such that x_1 is a non-zero divisor in R and x_i for $i > 0$ is a non-zero divisor in $R/(x_1, \dots, x_{i-1})$.

Lemma 4.6.13. Any minimal generating system $x_1, \dots, x_n$ of a the maximal ideal of a regular local ring R forms a regular sequence on R.

Proof. [17, Proposition 4.4.6] □

Lemma 4.6.14. Let R be a ring and $I \subset R$ a finitely generated ideal such that R/I is artinian. Then R/I^n is artinian for all $n \in \mathbb{N}$.

Proof. It is enough to show that R/I^n has finite length as R-module. This can be done via induction on n. In the induction step one considers the short exact sequence

$$0 \to I^n/I^{n+1} \to R/I^{n+1} \to R/I^n \to 0$$

and concludes as $l(R/I^{n+1}) = l(R/I^n) + l(I^n/I^{n+1})$. Here, $l(I^n/I^{n+1})$ is finite since I^n/I^{n+1} is a finitely generated R/I-module. □

Proposition 4.6.15 (Koszul resolution)**.** Let R be a ring and $I = (x_1, \dots, x_n)$ an ideal which is generated by a regular sequence. Then there is a free resolution of R/I given by

$$0 \to \Lambda^n(R^n) \to \cdots \to \Lambda^2(R^n) \to R^n \xrightarrow{x} R \to R/I \to 0.$$

Here, $R^n \xrightarrow{x} R$ is the morphism that comes from the multiplication with x_i in each component. The module $\Lambda^p(R^n)$ is finitely free of rank $\binom{n}{p}$. Applying sheafification to this sequence yields therefore a flat resolution of the constant sheaf with value R/I.

Proof. [17, Corollary 4.5.5] □

Lemma 4.6.16. Assume R is a ring and $I \subset R$ an ideal such that R/I is artinian. Let $(\epsilon^* F_n)_{n \in \mathbb{N}}$ be an inverse system such that for all $n \in \mathbb{N}$ we have $I^{n+1}F_n = 0$ and $F_n \in \mathrm{Mod}_R(X_{\text{et}})$ is locally constant with finitely generated values. Then the system $(\epsilon^* F_n)_{n \in \mathbb{N}}$ fulfills the Mittag-Leffler condition.

Proof. Assume $(F_n)_{n\in\mathbb{N}}$ is an inverse system of locally constant étale sheaves with finitely generated stalks. By passing to connected components, we can assume that X is connected. It is enough, to check the Mittag-Leffler condition at one stalk η, as the specilization maps give isomorphisms $F_{n\eta} \cong F_{nx}$ for any other geometric point x (see Proposition 2.2.3). The rings R/I^n are artinian by Lemma 4.6.14 and $F_{n\eta}$ is a finitely generated R/I^{n+1}-module. Now the claim is clear, as any finitely generated module over an artinian ring has the descending chain condition. $\square$

Lemma 4.6.17. Assume $K \in \mathrm{D}_{\mathrm{cons}}(X_{\mathrm{proet}}, R)$. Then there exists a finite stratification $X = \bigsqcup_{i=1}^{n} V_i$, such that the cohomology sheaves of $(K \otimes_R^{\mathrm{L}} R/I)|_{V_i}$ are given by the pullbacks of locally constant sheaves with finitely generated values on X_{et}.

Proof. By definition, $K \otimes_R^{\mathrm{L}} R/I$ is a bounded complex which is the pullback of an constructible complex on X_{et}. Hence, it is of the form

$$\cdots 0 \to \epsilon^* F^a \to \cdots \to \epsilon^* F^b \to 0 \cdots$$

for $a, b \in \mathbb{Z}$. Proposition 4.3.12 implies that vor every $k \in \{a, \dots, b\}$ there is a finite stratification $X = \bigsqcup_{i_k=1}^{n_k} V_{k,i_k}$ such that $(\epsilon^* F)|_{V_{k,i_k}}$ is the pullback of a locally constant étale sheaf with finitely generated values. The family

$$\left(\bigcap_{k=a}^{b} V_{k,i_k} \right)_{(i_a,\dots,i_b)}$$

is then a finite stratification which fulfills our claim. $\square$

Lemma 4.6.18. Let $K \in D^-(X_{\mathrm{proet}}, R)$ and let $V_i \hookrightarrow X$ be a locally closed immersion. We then have the formula

$$(K \otimes_R^{\mathrm{L}} R/I)|_{V_i} = K|_{V_i} \otimes_R^{\mathrm{L}} R/I.$$

Proof. Derived pullback commutes with derived tensor product, see [15, Tag 0D6D]. As by our conventions pullbacks are exact, the claim follows trivially from this fact. $\square$

Lemma 4.6.19. Assume $(F_n)_{n\in\mathbb{N}}$ is an inverse system in $\mathrm{Mod}_R(X_{\mathrm{proet}})$ which fulfills the Mittag-Leffler condition. Then

$$\mathrm{Rlim}_n(F_n) \cong \lim_n F_n[0]$$

Proof. The corresponding statement is true for R-modules for any ring R, see Lemma 4.1.4. Assume $U \in X_{\mathrm{proet}}$ is any weakly contractible object. As $\Gamma(U, -)$ is exact, we have $\mathrm{R}\Gamma(U, -) = \Gamma(U, -)$. Moreover, by Example 4.5.3 derived limits commute with $\mathrm{R}\Gamma(U, -)$. Hence,

$$\mathrm{R}\Gamma(U, -) \circ \mathrm{Rlim}_n(F_n) = \mathrm{Rlim}_n(\mathrm{R}\Gamma(U, F_n)) = \mathrm{Rlim}_n(F_n(U)).$$

But as U is weakly contractible, the system $(F_n(U))_{n\in\mathbb{N}}$ fulfills the Mittag-Leffler condition and therefore

$$\mathrm{Rlim}_n(F_n(U)) \cong \mathrm{lim}_n F_n(U).$$

This proves the claim as X_{proet} has enough weakly contractible objects. □

Lemma 4.6.20. Assume $(L_n)_{n\in\mathbb{N}}$ is a system of uniformly bounded complexes in $\mathrm{D}(X_{\mathrm{proet}}, R)$ such that $\mathrm{Rlim}_n(H^i(L_n)) = 0$ for all $i \in \mathbb{Z}$. Then $\mathrm{Rlim}_n(L_n) = 0$.

Proof. As $(L_n)_{n\in\mathbb{N}}$ is uniformly bounded, there is a $k \in \mathbb{Z}$ such that for all $n \in \mathbb{N}$ and all $i < k$ the cohomology sheaves $H^i(L_n)$ vanish. In particular, the truncation $\tau^{\leq k} L_n$ computes as

$$\tau^{\leq k} L_n = H^k(L_n)[k].$$

We can apply Rlim to the exact triangle

$$(\tau^{\leq k} L_n)_{n\in\mathbb{N}} \to (L_n)_{n\in\mathbb{N}} \to (\tau^{\geq k+1} L_n)_{n\in\mathbb{N}} \to (\tau^{\leq k} L_n)_{n\in\mathbb{N}}[-1].$$

By assumption $\mathrm{Rlim}_n(H^k(L_n)) = 0$, which implies $\mathrm{Rlim}_n(L_n) \cong \mathrm{Rlim}_n(\tau^{\geq k+1} L_n)$. Proceeding with $\tau^{\geq k+1} L_n$, one can inductively show $\mathrm{Rlim}_n(L_n) \cong \mathrm{Rlim}(\tau^{\geq k'} L_n)$ for any $k' \in \mathbb{N}$. As $(L_n)_{n\in\mathbb{N}}$ is uniformly bounded, this proves the claim. □

Theorem 4.6.21. Let R be a regular local ring with maximal ideal $I \subset R$. Further fix an $\mathcal{F} \in \mathrm{Mod}(X_{\mathrm{proet}}, R)$ and consider the complex $K := \mathcal{F}[0]$. Then $\mathcal{F}$ is a constructible R-sheaf[10] if and only if K is a constructible complex.

Proof. For the proof we fix the following notation:

$$K_n := K \otimes^{\mathrm{L}}_R R/I^n.$$

($\Leftarrow$) Assume K is a constructible complex. The constructibility of $\mathcal{F}/I^n\mathcal{F}$ is covered by Proposition 4.6.9. It remains to prove the completeness of $\mathcal{F}$. By Corollary 4.4.13, it is enough to show that $\mathcal{F}|_{V_i}$ is an I-adic sheaf for a finite stratification $X = \bigsqcup_{i=1}^n V_i$. Using Lemma 4.6.18 and Lemma 4.6.17, we can assume that K_1 is the pullback of a locally constant complex with finitely generated values. With the strategy of Proposition 4.6.9, it is an easy exercise to show that in this case the cohomology sheaves of K_n are the pullbacks of locally constant étale sheaves of finite type. Moreover, we know that $K \cong \mathrm{Rlim}_n(K_n)$, as K is derived complete.
Claim. For $i < 0$ we have $\mathrm{Rlim}_n(H^i(K_n)) = 0$.
As K_1 has finite tor dimension, the K_n are uniformly bounded, see [15, Tag 0942]. That is, there are $k, k' \in \mathbb{Z}$, such that for all $n \in \mathbb{N}$ and $l \notin [k, k']$ the cohomology sheaves $H^l(K_n)$ vanish. In particular, for $i < k$, the equation $\mathrm{Rlim}_n(H^i(K_n)) = 0$ is trivially fulfilled.

[10] I.e. an I-adic sheaf on X_{proet}.

Assume there is an $m < 0$ such that $\mathrm{Rlim}_n(H^i(K_n)) = 0$ for all $i < m$. Consider the exact triangle

$$(\tau^{\leq m-1}K_n)_{n\in\mathbb{N}} \to (K_n)_{n\in\mathbb{N}} \to (\tau^{\geq m}K_n)_{n\in\mathbb{N}} \to (\tau^{\leq m-1}K_n)_{n\in\mathbb{N}}[-1] \tag{4.6}$$

and apply Lemma 4.6.20 to prove that $\mathrm{Rlim}_n(\tau^{\leq m-1}K_n) = 0$. Applying Rlim to the exact triangle (4.6) yields $K = \mathrm{Rlim}_n(K_n) \cong \mathrm{Rlim}_n(\tau^{\geq m}K_n)$. As $m < 0$, we have $H^m(K) = 0$ and hence $0 = H^m(\mathrm{Rlim}_n(\tau^{\geq m}K_n))$. The usual computations with derived functors show

$$0 = H^m(\mathrm{Rlim}_n(\tau^{\geq m}K_n)) = \mathrm{lim}_n H^m(K_n).$$

This is, because for all $n \in \mathbb{N}$ and all $i < m$ the sheaves $H^i(\tau^{\geq m}K_n)$ vanish. As explained above, the sheaves $H^m(K_n)$ are given by the pullbacks of locally constant étale sheaves of finite type. Lemma 4.6.16 shows that $(H^m(K_n))_{n\in\mathbb{N}}$ fulfills the Mittag-Leffler condition. Finally, lim and Rlim agree for ML-systems by Lemma 4.6.19. This proves

$$0 = \mathrm{lim}_n H^m(K_n) = \mathrm{Rlim}_n(H^m(K_n)).$$

Now, $0 = \mathrm{Rlim}_n(H^i(K_n))$ for all $i < m+1$ and we can proceed inductively to prove that $\mathrm{Rlim}_n(H^i(K_n)) = 0$ for all $i < 0$.
The claim together with Lemma 4.6.20 implies that $\mathrm{Rlim}_n(\tau^{\leq -1}K_n) = 0$. In particular, with a similar exact triangle as Eq. (4.6) we can show

$$\mathrm{Rlim}_n(K_n) \cong \mathrm{Rlim}(\tau^{\geq 0}K_n) = \mathrm{Rlim}_n(\mathcal{F} \otimes_R R/I^n).$$

To complete the proof we apply Proposition 4.2.34 to get isomorphisms

$$K \cong \mathrm{Rlim}_n(K_n) \cong \mathrm{Rlim}_n(\mathcal{F} \otimes_R R/I^n) \cong (\mathrm{lim}_n \mathcal{F} \otimes_R R/I^n)[0].$$

This directly implies $\mathcal{F} \cong \mathrm{lim}_n \mathcal{F}/I^n\mathcal{F}$, which we wanted to show.

($\Rightarrow$) Conversely, assume $\mathcal{F}$ is a constructible R-sheaf, i.e. it is complete and the quotient sheaves $\mathcal{F}/I^n\mathcal{F}$ are constructible. By Lemma 4.6.4, $\mathcal{F}$ is already derived complete. It remains to show that $\mathcal{F} \otimes^{\mathrm{L}}_R R/IR$ has constructible cohomology sheaves and is of finite tor dimension. Note that the claim about the finite tor dimension is trivial as R is a regular local ring and has therefore finite global dimension, see [17, Theorem 4.4.6].
For the constructibility of $H^i(\mathcal{F} \otimes^{\mathrm{L}}_R R/IR)$ we use induction on the number of generators of I. If $I = (a)$ for a non-zero devisor $a \in R$, then one can compute, using the Koszul resolution for $R/(a)$, that

$$\mathcal{F} \otimes^{\mathrm{L}}_R R/IR = 0 \to \mathcal{F} \xrightarrow{\cdot a} \mathcal{F} \to 0. \tag{4.7}$$

In particular, $H^0(\mathcal{F} \otimes^{\mathrm{L}}_R R/IR) = \mathcal{F}/a\mathcal{F}$ and $H^1(\mathcal{F} \otimes^{\mathrm{L}}_R R/IR) = \ker(\mathcal{F} \xrightarrow{\cdot a} \mathcal{F}) =: \mathcal{F}[a]$. By the argumentation of Proposition 4.6.9, both cohomology sheaves are constructible and we are done.

Now assume $I = (a_1, \dots, a_n)$ for an $n \geq 2$. Define $R' := R/(a_1)$ and denote by $I' \subset R$ the ideal generated by $(a_2, \dots, a_n)$ in R'. The usual rules for computation with the derived tensor product imply

$$\mathcal{F} \otimes^{\mathrm{L}}_R R/IR \cong \left(\mathcal{F} \otimes^{\mathrm{L}}_R R/(a_1)\right) \otimes^{\mathrm{L}}_{R'} R'/I'.$$

The complex $C^* := \mathcal{F} \otimes^{\mathrm{L}}_R R/(a_1)$ looks similar to Eq. (4.7). We can compute the truncations

$$\tau^{\leq -1} C^* = 0 \to \mathcal{F}[a_1] \to 0 \to 0 \quad \text{and} \quad \tau^{\geq 0} C^* = 0 \to 0 \to \mathcal{F}/a_1\mathcal{F} \to 0$$

By Corollary 4.4.8, both $\mathcal{F}[a_1]$ and $\mathcal{F}/a_1\mathcal{F}$ are I-adic sheaves and, in particular, they are I'-adic sheaves of R'-modules. We have a canonical exact triangle

$$\tau^{\leq -1} C^* \to C^* \to \tau^{\geq 0} C^* \to \tau^{\leq -1} C^*[-1].$$

The induction hypothesis shows that $(\tau^{\leq -1} C^*) \otimes^{\mathrm{L}}_{R'} R'/I'$ and $(\tau^{\geq 0} C^*) \otimes^{\mathrm{L}}_{R'} R'/I'$ have constructible cohomology sheaves. Apply the derived functor $(-) \otimes^{\mathrm{L}}_{R'} R'/I'$ to the above exact triangle and consider the associated long exact sequence of cohomology sheaves. As the constructible sheaves on X_{proet} form a weak Serre subcategory by Corollary 4.3.10, the cohomology sheaves of $C^* \otimes^{\mathrm{L}}_{R'} R'/I' \cong \mathcal{F} \otimes^{\mathrm{L}}_R R/IR$ are constructible. This completes the proof.

□

4.7 Two Versions of a Pro-étale Site

4.7.1 The Pro-étale Site by B. Bhatt and P. Scholze

The goal of this section is to give a construction of the pro-étale site introduced in the paper [2] of B. Bhatt and P. Scholze. Their construction satisfies all desired axioms, thereby justifying that the theory presented above is valid. Indeed, a concrete description offers advantages over the axiomatic viewpoint. For instance, certain statements in the mentioned paper heavily rely on the explicit construction and may not be generalized to our setting. After giving the construction, we will state some of the properties, the pro-étale topology of Bhatt and Scholze fulfills and which may not generalize to the setting given in Section 4.3. First the notion of weakly étale morphisms is introduced. We also refer to the respective section on pro-étale cohomology in the Stacks Project [15, Tag 0965].

Definition 4.7.1. A morphism of rings $A \to B$ is called *weakly étale* if it is flat and if the morphism $B \otimes_A B \to B$ is flat.
Analougusly, a morphism of schemes $X \to Y$ is *weakly étale* if $X \to Y$ and $X \to X \times_Y X$ are flat.

Lemma 4.7.2. The composition of two weakly étale maps is again weakly étale. Further, the class of weakly étale morphisms is stable under base change. That is, if $X \to Y$ is a weakly étale morphism and $Y' \to Y$ is a morphism of schemes, then the base change map $X \times_Y Y' \to Y'$ is weakly étale. Moreover, if $f : X \to Y$ fits in a commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

where g and h are weakly étale then f is already weakly étale.

Proof. [2, Lemma 4.1.6] and [2, Lemma 4.1.7] □

Definition 4.7.3. Let X be a scheme. Define X_{proet} as the category of weakly étale morphisms over X. A family $(f_i : Y_i \to Y)_{i \in I}$ of weakly étale morphisms is defined to be a covering if it is a covering in the fpqc topology, i.e. for every affine open $U \subset Y$ there is a map $\alpha : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{\alpha(j)}$ such that $U = \bigcup_{i=1}^{n} f_{\alpha(i)}(V_i)$. That these coverings form a topology on X_{proet} can be found in [2, §4.1].

It is not part of this thesis to go into deep calculations with this particular definition. Nevertheless, we want to provide references to demonstrate that the pro-étale site fulfilles all the desired axioms. We will also explain the parts that are not directly accessible in the literature. For this section fix a scheme X.

Lemma 4.7.4. The pro-étale topology on a scheme X is finer than the étale topology. That is, any étale covering is also a pro-étale covering.

Proof. [15, Tag 098B] □

Definition 4.7.5. An object $U \in X_{\text{proet}}$ is called *pro-étale affine* if there is a small cofiltered diagram $(U_i)_{i \in I}$ of affine étale schemes over X such that $U = \lim_i U_i$. The full subcategory of X_{proet} spanned by pro-étale affines is denoted $X^{\text{aff}}_{\text{proet}}$.

Lemma 4.7.6. The topos $\mathrm{Sh}(X_{\text{proet}})$ is generated by $X^{\text{aff}}_{\text{proet}}$. That means that any $Y \in X_{\text{proet}}$ can be covered by elements in $X^{\text{aff}}_{\text{proet}}$.

Proof. [2, Lemma 4.2.4] □

Lemma 4.7.7. If X is an affine scheme, then $X^{\text{aff}}_{\text{proet}}$ is simply the category of all affine schemes which are pro-étale over X.

Proof. [2, Remark 4.2.5] □

Lemma 4.7.8. Assume X is affine and $i : Z \hookrightarrow X$ is a closed immersion. Then the functor

$$\begin{aligned} i^{-1} : X^{\text{aff}}_{\text{proet}} &\longrightarrow Z^{\text{aff}}_{\text{proet}} \\ Y &\longmapsto Y \times_X Z \end{aligned}$$

has a fully faithful left adjoint $V \mapsto \tilde{V}$. Moreover, for any weakly contractible $V \in Z^{\text{aff}}_{\text{proet}}$ and any sheaf $F \in \text{Sh}(X_{\text{proet}})$ one has

$$i^*F(V) = F(\tilde{V}).$$

Proof. [2, Lemma 6.1.1] and [2, Lemma 6.1.3]. □

Definition 4.7.9. Any étale morphism is weakly étale and any étale covering is a pro-étale covering. Hence, the inclusion $X_{\text{et}} \hookrightarrow X_{\text{proet}}$ defines a morphism of sites $\epsilon : X_{\text{proet}} \to X_{\text{et}}$. It is obvious that ϵ defines a natural transformation $(-)_{\text{proet}} \to (-)_{\text{et}}$.

Lemma 4.7.10. Let $i : Z \to X$ be a closed immersion with quasi-compact open complement U. Then i_* preserves surjections. Moreover, if $V \in X_{\text{proet}}$ is weakly contractible then $V_0 := V \times_X Z$ is weakly contractible.

Proof. Assume $\phi : F \to G$ is a surjective map of sheaves on Z_{proet}. Let $Y \in X_{\text{proet}}$ and $g \in i_*G(Y)$. Define $Y_0 := Y \times_X Z$ and $Y|_U := Y \times_X U$. Then there exists a cover $(W \to Y_0)$ by weakly contractible objects such that $g|_W$ is in the image of ϕ. The quasi-compactness of U implies that $\tilde{W} \sqcup Y|_U \to Y$ is a cover[11] of Y such that $G(Y|_U) = F(Y|_U) = \{*\}$ and $g|_{\tilde{W}}$ is in the image of $i_*\phi(\tilde{W})$. This trivially implies that $i_*F \to i_*G$ is surjective. The second claim follows from the first by a straightforward calculation. □

The main focus of this section lies on the following theorem, using several properties of Section 4.4.

Theorem 4.7.11. The pro-étale topology from Definition 4.7.3 is a pro-étale enlargement for $E = \text{Sch}$. In particular, the pro-étale site fulfilles all axioms from Definition 4.3.1.

Proof. For most of the statements we will give references. However, Axiom 4 is not readily available in [2], so we will give some ideas for this particular axiom.

In X_{proet} any object can be covered by affine weakly contractible objects, so Axiom 1 is true. This fact can be found in [15, Tag 0F4P]. Axiom 2 is the statement of [15, Tag 099V].

For any étale morphism $U \to X$, we have $X_{\text{proet}}/U = U_{\text{proet}}$ by definition. So Axiom 3 is trivially fulfilled.

To prove Axiom 4 we will use the ideas from [2, Section 6.1]. Let $i : Z \hookrightarrow X$ be a closed immersion with quasi-compact open complement $j : U \hookrightarrow X$. Assume G is a sheaf of sets on X_{proet} such that $G \times h_U \xrightarrow{\cong} h_U$ is an isomorphism. We have to show that $G \cong i_*i^*G$. As this can be checked locally, we pick a weakly contractible $V \in X^{\text{aff}}_{\text{proet}}$. Define $V_0 := V \times_X Z \in Z^{\text{aff}}_{\text{proet}}$ and $V|_U := V \times_X U \in X^{\text{aff}}_{\text{proet}}$. Note that Lemma 4.7.10 implies that V_0 is a weakly contractible affine object in Z_{proet}. As U is quasi-compact,

[11] The quasi-compactness ensures that the cover is a cover in the fpqc topology.

we deduce that $V|_U$ is quasi-compact and hence it is true that $\tilde{V_0} \sqcup V|_U \to V$ is a cover of V. We can finally compute

$$i_*i^*G(V) = i^*G(V_0) = G(\tilde{V_0}).$$

As $V|_U$ factorizes over U, we can use the equality $G \times h_U \xrightarrow{\cong} h_U$ to show $G(V|_U) = \{*\}$ and $G(V|_U \times_X \tilde{V_0}) = \{*\}$. The sheaf property applied to the cover $\tilde{V_0} \sqcup V|_U \to V$ implies that $G(V) \cong G(\tilde{V_0}) \cong i_*i^*G(V)$, which proves $i_*i^*G \cong G$.
Conversely assume $i_*i^*G \cong G$, then for any object $V \in X_{\text{proet}}$, which factorizes over U, we have $i_*i^*G(V) \cong i^*G(V \times_X Z) = i^*G(\emptyset) = \{*\}$. This means, $G \times h_U \xrightarrow{\cong} h_U$ is an isomorphism.

Finally, Axiom 5 can be found in [2, Corollary 6.1.5]. For instance, the statement without additional assumptions on the closed immersion i can be referenced in [15, Tag 09BL]. □

Facts 4.7.12. Here is a list of some facts which are true in the pro-étale topology of B. Bhatt and P. Scholze and which might not generalize to our setting.

- The site X_{proet} is subcanonical.
- For any $Y \in \text{Sh}(X_{\text{proet}})$ we have $\text{Sh}(Y_{\text{proet}}) \cong \text{Sh}(X_{\text{proet}})/Y$. This is indeed an improvement to Axiom 3. In our definition, this is only demanded for open immersions $U \hookrightarrow X$.
- Being classical can be checked on a pro-étale cover $(X_i \to X)_{i \in I}$. That is, an $F \in \text{Sh}(X_{\text{proet}})$ is the pullback of an étale sheaf if and only if $F|_{X_i}$ is the pullback of an étale sheaf for all $i \in I$.
- If $f : Y \to X$ is a quasi-compact and quasi-separated map of schemes and $F \in \text{Sh}(Y_{\text{et}})$, then the canonical morphism

$$\epsilon^* f_{\text{et}*}F \to f_{\text{proet}*}\epsilon^* F$$

  is an isomorphism. Our theory gives this result only for a closed immersion with quasi-compact complement, see Lemma 4.3.11.
- The pushforward f_* for a finitely presented and finite morphism of schemes $f : Y \to X$ is exact.

4.7.2 The Pro-étale Site by M. Kerz

In this short section we want to indicate that there is another possibility to define a site which fulfills all the axioms from Section 4.3. We only give a rough idea and sketch the construction. We completely omit the proofs.

Definition 4.7.13 (pro-categories)**.** Let $\mathcal{C}$ be a category. Then we define the *pro-category* pro-$\mathcal{C}$ associated to $\mathcal{C}$ as follows

Objects are functors $I \to \mathcal{C}$, where I is a small cofiltered category.

Morphisms are given by $\mathrm{Mor}_{\text{pro-}\mathcal{C}}(F,G) := \lim_j \mathrm{colim}_i \mathrm{Mor}_{\mathcal{C}}(F(i),G(j))$.

Definition 4.7.14. Let λ be a ordinal and $\mathcal{C}$ a category. A functor $F : \lambda^{\mathrm{opp}} \to \mathcal{C}$ is called *λ-tower* if for any limit ordinal $\mu < \lambda$ the limit

$$F_{<\mu} := \lim_{i<\mu} F_i$$

exists and the canonical morphism $F_\mu \to F_{<\mu}$ is an isomorphism.

Definition 4.7.15. The *transfinite topology on* pro-$\mathcal{C}$ is the coarsest topology on pro-$\mathcal{C}$ such that

1. The canonical functor $\mathcal{C} \to$ pro-$\mathcal{C}$ is continuous.
2. Assume $(F_i)_{i<\lambda}$ is a tower in pro-$\mathcal{C}$ such that the maps $F_{i+1} \to F_i$ are covering morphisms. Then the canonical morphism $\lim_{i<\lambda} F_i \to F_0$ is a covering morphism.

Definition 4.7.16. Let X be a quasi-compact and separated scheme. Note that the site X_{et} is then admissible. Define X_{proet} as the site given by pro-X_{et} together with the transfinite topology. The morphism of sites $\epsilon : X_{\mathrm{proet}} \to X_{\mathrm{et}}$ comes from the canonical functor $X_{\mathrm{et}} \to$ pro-X_{et}, which sends an étale scheme U over X to the functor $\{*\} \to X_{\mathrm{et}}$ where $* \mapsto U$.

Theorem 4.7.17. Definition 4.7.16 defines a pro-étale enlargement for the category E of quasi-compact and separated schemes.

Proof. The functoriality results follow immediately from the definitions. They are left as an exercise. Axiom 1 and Axiom 2 are covered by [12, Theorem 4.2] and [12, Proposition 6.6], respectively. Note that the notion of weakly contractible objects in the sense of [12] agrees with our definition by the observations in Remark 4.2.26.
For Axiom 3 consider an étale scheme U over X. We will also write U for $\epsilon^{-1}(U) \in X_{\mathrm{proet}}$. By definition we have

$$\mathrm{Mor}_{\text{pro-}X_{\mathrm{et}}}(F,U) = \mathrm{colim}_j \mathrm{Mor}_{X_{\mathrm{et}}}(F(i),U).$$

In particular, a morphism $\phi : F \to U$ in the category pro-X_{et} is the same as a functor $\phi : I \to U_{\mathrm{et}}$ such that F is the composition $I \xrightarrow{\phi} U_{\mathrm{et}} \to X_{\mathrm{et}}$. Hence, pro-$U_{\mathrm{et}}$ and pro-X_{et}/U are canonically isomorphic.
Axiom 4 for a closed immersion $i : Z \hookrightarrow X$ works similarly as the respective part in Theorem 4.7.11. If X if affine, [12, Lemma 10.5] gives a concrete left adjoint $i_\flat^{\mathrm{pre}}$ to the functor

$$(-) \times_X Z : X_{\mathrm{proet}} \to Z_{\mathrm{proet}}$$

Moreover, the presheaf pullback $i^\bullet$ is given by the formula,

$$i^\bullet F(V) = F(i_\flat^{\mathrm{pre}} V).$$

The proof of [15, Tag 09BL] shows that for a weakly contractible object V we have $i^* F(V) = i^\bullet F(V)$. We can use the strategy of Theorem 4.7.11 to prove the claim. Finally, Axiom 5 is covered by [12, Section 10]. $\square$

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