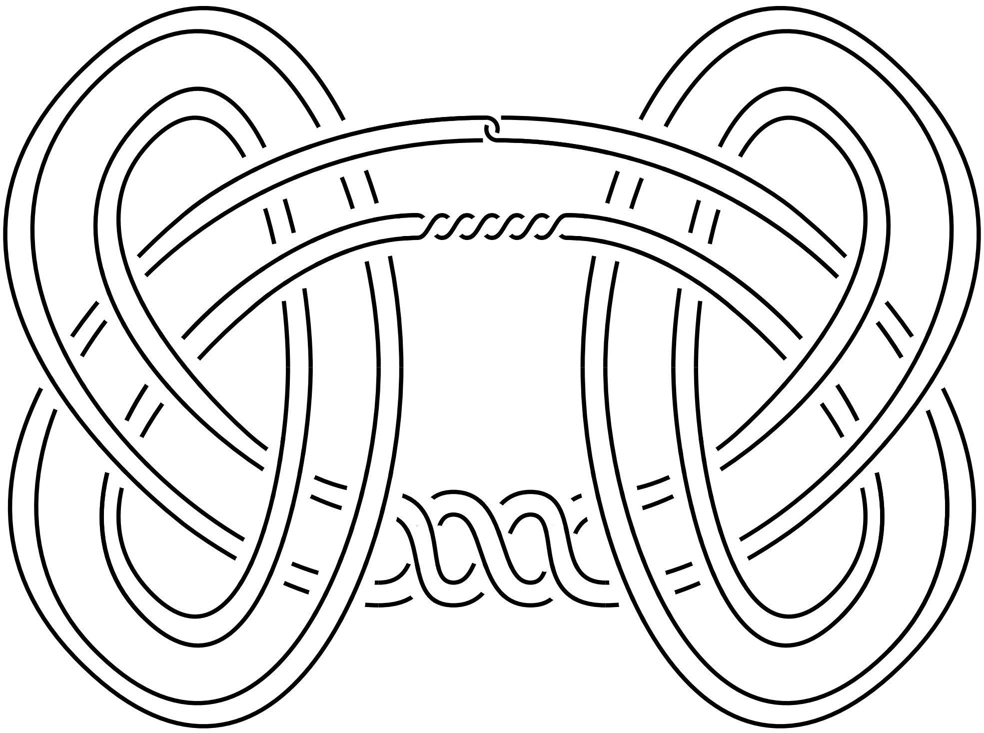
The Alexander polynomial of Satellite Knots

Bachelors thesis in Mathematics
submitted by

Tobias Hirsch



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Contents

Abstract

In the first chapter of this thesis we define the Alexander polynomial of a knot using the Seifert matrix of the knot and show that it is a knot invariant. For this we introduce the linking number of two knots and recall Seifert surfaces. Exemplarily, we compute the Alexander polynomial for certain knots and prove some of its properties.
Then, in the second chapter, we define satellite knots and sketch their basic theory. As examples we consider cable knots, the connected sum of knots and the Whitehead double of a knot. The main result of this work is to give a formula for the Alexander polynomial of a satellite knot.
We encounter torus knots when defining cable knots. In the last chapter we determine their Alexander polynomial and classify them based on it.

Preliminaries

Literature

No proofs in this thesis are directly copied from the literature. In the first chapter we mainly follow the account in [Rol03, Chapter 8] and [Lic97, Chapter 6]. The proof of the main theorem expands upon [Lic97, Theorem 6.15]. The last chapter is based on ideas from [Mur08, Chapter 7]. The proofs used therein have not been provided by the literature.
We have used the table of knot invariants provided by [LM23] to find knots with specific properties.

Definitions and conventions

We view S^3 as the unit sphere in $\mathbb{C}^2$ and fix some orientation on it. For drawing, we remove a point and view $S^3 \setminus \{*\}$ as $\mathbb{R}^3$ in such a way that the orientation is given by the standard orientation on $\mathbb{R}^3$.
An m-component link is a closed 1-dimensional smooth submanifold of S^3 with $m \in \mathbb{N}$ components. A knot is a connected link. A link is oriented if it is oriented as a submanifold.
We say an m-component link L is trivial or an unlink if it bounds m disjoint smoothly embedded discs $\overline{B}^2$ in S^3.
Two m-component links L, J are equivalent, if there exists a smooth isotopy

$$F\colon \left(\bigsqcup_{i=1}^{m} S_1\right) \times [0,1] \to S^3$$

with $L = \mathrm{im}(F_0)$ and $J = \mathrm{im}(F_1)$. If L and J are oriented, we additionally demand that $F_1 \circ F_0^{-1}\colon L \to J$ is orientation preserving.
For an (oriented) link L we denote by L^* its mirror image and by $\overline{L}$ the link with the orientation reversed on all components.

1 Construction of the Alexander polynomial

1.1 Linking numbers

We begin with a section on the linking number of knots in S^3 exploring some of its key properties for later use.

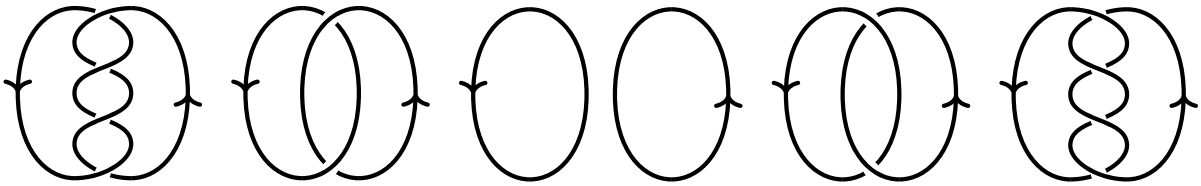
Definition. Let $J, K \subseteq S^3$ be disjoint oriented knots. We choose a diagram D for the link $J \cup K \subseteq S^3$ and consider

positive crossing negative crossing

Then the *linking number of J and K* is defined as

$$\mathrm{lk}(J, K) := \#\{\text{positive crossings in } D\} - \#\{\text{negative crossings in } D\}$$

Example 1.1.



$\mathrm{lk} = -2$ $\mathrm{lk} = -1$ $\mathrm{lk} = 0$ $\mathrm{lk} = 1$ $\mathrm{lk} = 2$

The following lemma gives some basic properties of linking numbers. In particular, it shows that the linking number is a well-defined link invariant.

Lemma 1.2. *Let $J, K \subseteq S^3$ be disjoint oriented knots. The linking number $\mathrm{lk}(J, K)$ is an invariant of the link $J \cup K$ and is commutative*

$$\mathrm{lk}(J, K) = \mathrm{lk}(K, J).$$

Furthermore, reversing the orientation of one knot or mirroring both knots changes the sign:

$$\mathrm{lk}(J, K) = -\,\mathrm{lk}(J, \overline{K}) = -\,\mathrm{lk}(\overline{J}, K) = -\,\mathrm{lk}(J^*, K^*)$$

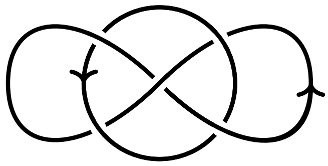
Proof. By the Reidemeister move theorem (see [OSS15, Theorem B.1.1]) it suffices to prove that the linking number is invariant under Reidemeister moves:

- A Reidemeister I move only introduces or removes a crossing of J (or K) with itself, hence it leaves the number of positive and negative crossings unchanged.
- A Reidemeister II move either leaves the number of positive and negative crossings unchanged (if it only affects one of the knots, or if it moves J over K) or introduces/removes exactly one positive crossing and one negative crossing, leaving $n_1 - n_2$ unchanged.
- A Reidemeister III move leaves the number of positive and negative crossings unchanged, as it just changes the position of crossings, but not their type.

If we change all crossings in a diagram D for $J \cup K$ we obtain a diagram for the mirror link of $J \cup K$. Reflecting the result along a line gives us again a diagram D' for $J \cup K$. But we now have turned a crossing of K over J in D into a crossing of J over K in D' without affecting the sign of the crossing. Using D to calculate $\mathrm{lk}(J, K)$ and D' to calculate $\mathrm{lk}(K, J)$ therefore shows commutativity.
Reversing the orientation of one of the two knots, turns a positive crossing into a negative one and vice versa. Therefore it changes the sign of the linking number.
Mirroring both J and K turns a positive crossing of K over J into a negative crossing of J over K. This changes the sign of the linking number, since the linking number is commutative. ■

We have now proven that the linking number is a link invariant, thus we can use it to show that two knots are actually linked. Unfortunately, there are 2-component links with linking number 0 that are not an unlink:

Example 1.3. The two knots forming the Whitehead link below have linking number 0.



We will later see in Example 1.22 that they are non-trivially linked.

There are many different ways to define the linking number of two knots. A list of eight possible definitions can be found in [Rol03, 5.D. Linking numbers]. The next lemma directly follows by choosing definition (1) in that list:

Lemma 1.4. *Let $J, K \subseteq S^3$ be disjoint oriented knots and let μ be a knot that generates $\mathrm{H}_1(S^3 \setminus J) \cong \mathbb{Z}$. There exists some $n \in \mathbb{Z}$ such that $[K] = n \cdot [\mu]$ and $\mathrm{lk}(J, K) = \mathrm{lk}(J, \mu) \cdot n$.*

1.2 Seifert surfaces

The goal of this section is to define the Seifert matrix of a knot and study how we can turn it into a knot invariant. We begin by recalling Seifert surfaces, see [Lic97, Chapter 2]:

Definition. Let $K \subseteq S^3$ be a knot or link. A *Seifert surface for K* is a compact orientable connected 2-dimensional smooth submanifold $\Sigma \subseteq S^3$ such that $\partial\Sigma = K$. If K is oriented, we demand that Σ is oriented such that the orientation on $K = \partial\Sigma$ is given by the boundary orientation.

Theorem 1.5. *Every (oriented) link $K \subseteq S^3$ has a Seifert surface.*

Definition. Let $K \subseteq S^3$ be a knot or link. The *genus of K* is the minimal genus of a Seifert surface for K.

Furthermore, we consider the following definition and theorem:

Definition. Let $Y \subseteq X$ be topological spaces. An embedding $\beta \colon Y \times [-1, 1] \to X$ such that $\beta(y, 0) = y$ for all $y \in Y$ is a *bicollar of Y in X*. For every subset $A \subseteq Y$ we refer to $A^+ := \beta(A \times \{1\})$ as the *positive push-off of A*.

Theorem 1.6. *Let M be an oriented smooth manifold and $N \subseteq M$ be a compact oriented codimension-1 smooth submanifold. Then N has an orientation preserving smooth bicollar in M and any two orientation preserving smooth bicollars of N in M are smoothly isotopic rel N.*

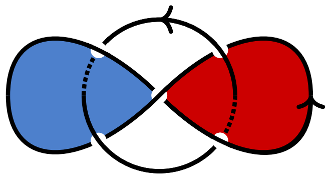
Proof. A bicollar may essentially be treated as a codimension-1 tubular neighbourhood as defined in [Wal16, Chapter 2.3]. Existence then follows from [Wal16, Theorem 2.3.8] and the uniqueness from [Wal16, Theorem 2.5.5, Proposition 2.5.8]. ■

Having reminded the reader of Seifert surfaces, we can now introduce another definition for the linking number of knots from [Rol03, 5.D. Linking numbers]:

Lemma 1.7. *Let $J, K \subseteq S^3$ be disjoint oriented knots and let Σ be a Seifert surface for K with an orientation preserving smooth bicollar $\beta\colon \Sigma \times [-1,1] \to S^3$. We may assume that J intersects Σ transversally (see* [Wal16, Chapter 4.5]*), in particular, that $J \cap \Sigma$ is finite. We say an intersection point $x \in J \cap \Sigma$ is positive, if J passes from $\beta(\Sigma \times [-1,0])$ to $\beta(\Sigma \times [0,1])$ in a small neighbourhood of x, else we say the intersection point is negative. Then*

$$\mathrm{lk}(J,K) = \#\{\text{positive intersection points}\} - \#\{\text{negative intersection points}\}$$

Example 1.8.



We orient this surface such that the red side faces upwards. Then the left intersection is negative and the right intersection positive, showing again that the Whitehead link has linking number 0.

To a Seifert surface we can associate a bilinear form on its first homology group, which will be represented by the Seifert matrix. In detail, this turns out to be quite tricky to precisely define:

Construction 1.9. Let $\Sigma \subseteq S^3$ be a Seifert surface for an oriented knot or link K. By Theorem 1.6 there exists an orientation preserving smooth bicollar $\beta\colon \Sigma \times [-1,1] \to S^3$.
By the classification of surfaces, we can express every homology class $h \in \mathrm{H}_1(\Sigma)$ as an integer linear combination of smoothly embedded circles $\hat{h}_1, \ldots, \hat{h}_{n_h} \subseteq \Sigma$. We fix such a representation for every $h \in \mathrm{H}_1(\Sigma)$:

$$h = \sum_{i=1}^{n_h} k_{h_i} \cdot [\hat{h}_i]$$

By generalizing linking numbers to homology classes in the second entry via Lemma 1.4 we construct a well-defined map that is linear in the second entry:

$$\begin{aligned} \Phi\colon\ \mathrm{H}_1(\Sigma) \times \mathrm{H}_1(\Sigma) &\to \mathbb{Z} \\ (a,b) &\mapsto \sum_{i=1}^{n_a} k_{a_i} \cdot \mathrm{lk}(\hat{a}_i, b^+) \end{aligned}$$

This is independent of the choice of representation for a by Lemma 1.4, since

$$\sum_{i=1}^{n_a} k_{a_i} \cdot \mathrm{lk}(\hat{a}_i, b^+) = \sum_{i=1}^{n_a}\sum_{j=1}^{n_b} k_{a_i}k_{b_j} \cdot \mathrm{lk}(\hat{a}_i, \hat{b}_j^+) \overset{\text{Lemma 1.2}}{\overset{\downarrow}{=}} \sum_{i=1}^{n_a}\sum_{j=1}^{n_b} k_{a_i}k_{b_j} \cdot \mathrm{lk}(\hat{b}_j^+, \hat{a}_i) = \sum_{j=1}^{n_b} k_{b_j} \cdot \mathrm{lk}(\hat{b}_j^+, a)$$

Similarly it again follows from Lemma 1.2 that Φ is linear in the first entry and thereby a bilinear form on the $\mathbb{Z}$-module $\mathrm{H}_1(\Sigma)$.

In light of this construction, we define:

Definition. Let $K \subseteq S^3$ be a knot or link and $\Sigma \subseteq S^3$ be a Seifert surface for K. We choose an orientation preserving smooth bicollar $\beta\colon \Sigma \times [-1,1] \to S^3$. By Construction 1.9 there exists a bilinear form $\Phi\colon\ \mathrm{H}_1(\Sigma) \times \mathrm{H}_1(\Sigma) \to \mathbb{Z}$ such that for knots $a, b \subseteq \Sigma$

$$\Phi(a,b) = \mathrm{lk}(a, b^+)$$

For any basis $\{e_1, \ldots, e_n\}$ of $\mathrm{H}_1(\Sigma)$ as a $\mathbb{Z}$-module, we call the matrix

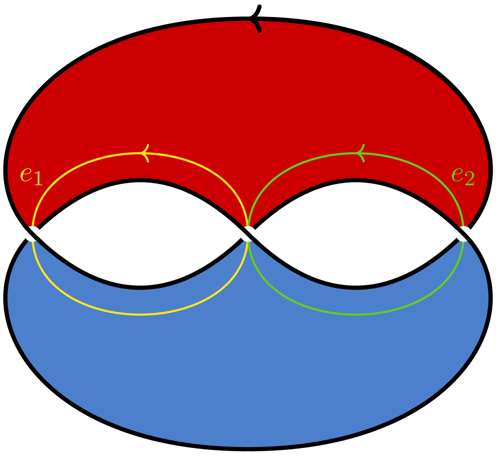
$$(\Phi(e_i, e_j))_{1 \le i,j \le n}$$

representing the Seifert form relative to this basis a *Seifert matrix for K*.

Example 1.10. Let $n \in \mathbb{N}$. A Seifert surface for the n-component unlink is given by a disk $\overline{D} \subseteq \mathbb{R}^2 \times \{0\} \subseteq S^3$ with $n-1$ open disks removed from its interior. We choose the obvious bicollar $\beta\colon \overline{D} \times [-1,1] \to \mathbb{R}^2 \times [-1,1] \subseteq S^3$. Then the Seifert form is zero. Hence, the Seifert matrix is an $(n-1) \times (n-1)$ zero matrix.

It is apparent that the Seifert matrix does not only depend on the knot, but also on the choice of Seifert surface and basis. This dependency is illustrated by the next example:

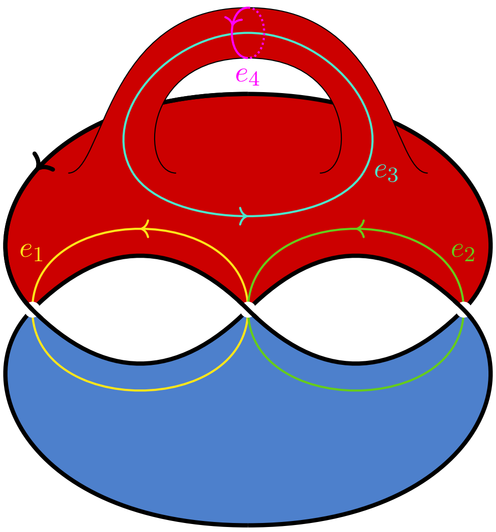
Example 1.11. A Seifert surface for the trefoil given by



The red side faces upwards by our orientation convention. The Seifert matrix for the trefoil originating from this Seifert surface relative to the basis $\{e_1, e_2\}$ is then given by

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

We can also consider a different Seifert surface for the trefoil, given by adding a tube to the Seifert surface above.



Considering the basis $\{e_1, \dots, e_4\}$ yields the Seifert matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To deduce a knot invariant from the Seifert matrix, we will have to study its dependency on the different choices. For the choice of basis, this is achieved purely by linear algebra:

Lemma 1.12. *Let $K \subseteq S^3$ be a knot or link and $\Sigma \subseteq S^3$ be a Seifert surface for K with an orientation preserving smooth bicollar $\beta\colon \Sigma \times [-1,1] \to S^3$. Then the Seifert matrix only depends on the choice of basis up to congruency over $\mathbb{Z}$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be bases for $H_1(\Sigma)$ and $M_{\mathcal{A}}$ (respectively $M_{\mathcal{B}}$) be the Seifert matrix relative to $\mathcal{A}$ (respectively $\mathcal{B}$). Because $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$ represent the same bilinear form (i.e. the Seifert form originating from Σ and β)

$$M_{\mathcal{A}} = S^T \cdot M_{\mathcal{B}} \cdot S$$

where S is the change-of-basis matrix from $\mathcal{A}$ to $\mathcal{B}$. ■

By considering some theory about bicollars one can also eliminate the dependence on the bicollar:

Lemma 1.13. *Let $K \subseteq S^3$ be a knot or link and $\Sigma \subseteq S^3$ be a Seifert surface for K. The Seifert form does not depend on the choice of orientation preserving smooth bicollar.*

Proof. Let $\beta_0, \beta_1\colon \Sigma \times [0,1] \to S^3$ be orientation preserving smooth bicollars for Σ. By Theorem 1.6 β_0 and β_1 are smoothly isotopic rel Σ in S^3 since they are orientation preserving bicollars. This implies that the Seifert forms of Σ with respect to β_0 and β_1 are identical, since the linking number is invariant under smooth isotopy by Lemma 1.2. ■

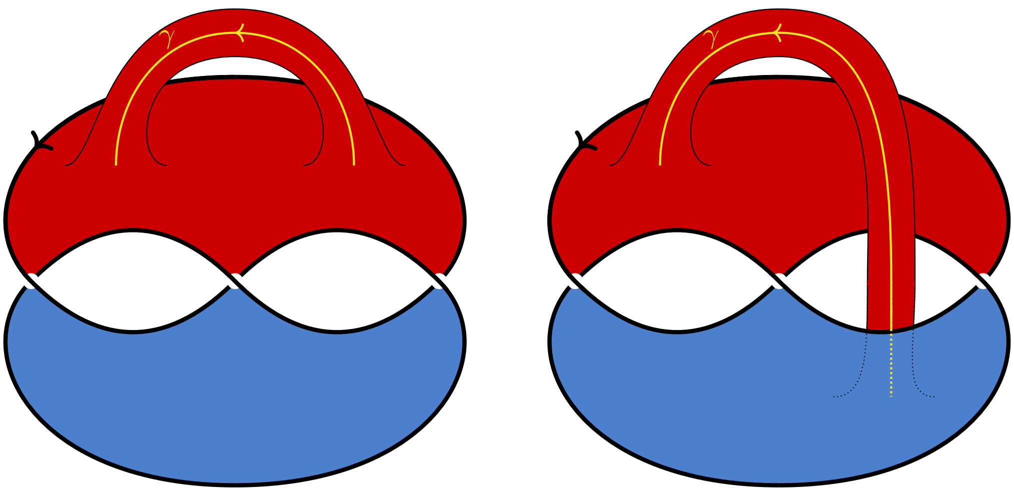
The last remaining dependency in the Seifert matrix – other than the knot of course – is the choice of Seifert surface. Therefore, we have to study the relationship between two Seifert surfaces for a given knot:

Definition. Let $\Sigma \subseteq S^3$ be a compact oriented 2-dimensional smooth submanifold and let $P, Q \in \Sigma \setminus \partial\Sigma$ be two distinct points. We consider a smooth embedding $\gamma\colon [0,1] \to S^3$ such that $\gamma(0) = P, \gamma(1) = Q$, $\gamma((0,1)) \cap \Sigma = \varnothing$ and

- a positive basis of $T_P\Sigma$ together with $\gamma'(0)$ form a positive basis of S^3
- a positive basis of $T_Q\Sigma$ together with $\gamma'(1)$ form a negative basis of S^3.

By deleting small disks around P and Q and appropriately gluing in a cylinder along γ, we obtain new compact oriented 2-dimensional smooth submanifold $\Sigma' \subseteq S^3$. Then Σ' is called a *stabilization of* Σ and Σ is called a *destabilization of* Σ'.[1]

Example 1.14.



stabilizations along γ

[1] Unfortunately, this definition has to remain rather vague, as a precise statement would be too lengthy to present here. We hope that – together with the accompanying figure – the situation becomes clear enough. Else, we refer the reader to [OSS15, Appendix B.3] for a slightly more detailed presentation.

Theorem 1.15 (Reidemeister–Singer). *Any two Seifert surfaces of an oriented link in S^3 become ambient isotopic after an appropriate sequence of stabilizations and destabilizations.*

Proof. see [OSS15, Appendix B.3]. The idea of the proof is as follows:
We first show that any Seifert surface can be stabilized to be the result of applying Seifert's algorithm to a diagram. Then we only have to prove that two Seifert surfaces resulting from Seifert's algorithm are related by stabilizations and destabilizations. This can be done by studying Reidemeister moves on the underlying diagrams. ∎

We now have the means necessary to describe the dependency of the Seifert matrix on the Seifert surface:

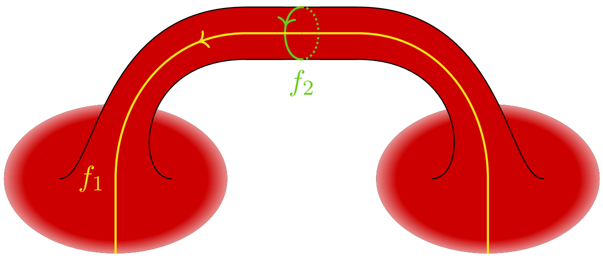
Definition. Let M be a square integer matrix and

$$M' := \begin{pmatrix} M & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad M' := \begin{pmatrix} M & 0 & 0 \\ * & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then M' is called an *elementary enlargement of* M and M is an *elementary reduction of* M'.
We say that two square matrices over $\mathbb{Z}$ are *S-equivalent*, if they are related by a sequence of elementary reductions, elementary enlargements and congruences over $\mathbb{Z}$.

Lemma 1.16. *Any Seifert matrices for a fixed oriented knot or link are S-equivalent.*

Proof. Let K be an oriented knot or link and M be a Seifert matrix for K originating from a Seifert surface Σ with homology basis $\{e_1, \dots, e_n\} \subseteq \mathrm{H}_1(\Sigma)$ and bicollar $\beta \colon \Sigma \times [-1,1] \to S^3$. Let M' be another Seifert matrix for K originating from a Seifert surface Σ'. By the Reidemeister-Singer Theorem 1.15 Σ and Σ' are ambient isotopic after a sequence of stabilizations and destabilizations. By Lemma 1.13 the Seifert form remains unchanged after ambient isotopy. By Lemma 1.12 we can therefore assume that Σ and Σ' are directly related by a sequence of stabilizations and destabilizations after applying a congruency over $\mathbb{Z}$ to M'. Therefore is it enough to prove that up to congruency over $\mathbb{Z}$ a stabilization of Σ corresponds to an elementary enlargement of M, and a destabilization of Σ corresponds to an elementary reduction of M:
Let Σ' be an elementary enlargement of Σ and let f_1 and f_2 be the 1-cycles indicated below. It is possible to connect the ends of f_1 since Σ is connected. We take any of the possible connections.



Then $\{e_1, \dots, e_n, f_1, f_2\} \subseteq \mathrm{H}_1(\Sigma')$ is a basis by a Mayer–Vietoris argument. We notice that f_2 bounds a disc in the cylinder glued in by the stabilization, hence by Lemma 1.4 for $i \in \{1, \dots, n\}$

$$\mathrm{lk}(f_2, e_i^+) = \mathrm{lk}(e_i, f_2^+) = \mathrm{lk}(f_2, f_2^+) = 0$$

Depending on the orientation of the bicollar in the cylinder either

$$\mathrm{lk}(f_1, f_2^+) = \pm 1,\ \mathrm{lk}(f_2, f_1^+) = 0 \qquad \text{or} \qquad \mathrm{lk}(f_1, f_2^+) = 0,\ \mathrm{lk}(f_2, f_1^+) = \pm 1$$

Therefore

$$M' = \begin{pmatrix} M & * & 0 \\ * & * & \pm 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad M' = \begin{pmatrix} M & * & 0 \\ * & * & 0 \\ 0 & \pm 1 & 0 \end{pmatrix}$$

which is congruent over $\mathbb{Z}$ to an elementary enlargement of M.
Applying the same process in reverse shows that destabilization of Σ leads to an elementary reduction of M. ∎

Before introducing the main character of this thesis, we explore the change of the Seifert matrix when reversing the orientation or taking mirror images for later reference:

Proposition 1.17. *Let $K \subseteq S^3$ be a knot or link with Seifert matrix M.*

- *A Seifert matrix for the reversion $\overline{K}$ is given by M^T*
- *A Seifert matrix for the mirror knot K^* is given by $-M^T$.*

Proof. Let Σ be a Seifert surface for K. By the classification of surfaces there exists a homology basis $\{e_1, \dots, e_n\} \subseteq \mathrm{H}_1(\Sigma)$ consisting of smoothly embedded circles. By Lemma 1.12 it suffices to prove the claim for a choice of basis, hence we may assume that M is the Seifert matrix for K resulting from Σ and the basis $\{e_1, \dots, e_n\} \subseteq \mathrm{H}_1(\Sigma)$.
For any two knots $a, b \subseteq \Sigma$ are the links $a \cup \beta(b \times \{1\}), \beta(a \times \{-1\}) \cup \beta(b \times \{1\}$ and $\beta(a \times \{-1\}) \cup b)$ smoothly isotopic through the bicollar. We therefore note that by Lemma 1.2

$$(*) \qquad \mathrm{lk}(a, \beta(b \times \{1\})) = \mathrm{lk}(\beta(a \times \{-1\}), \beta(b \times \{1\})) = \mathrm{lk}(\beta(a \times \{-1\}), b)$$

After this preliminary discussion we return to the original claims:

- Let $\overline{\Sigma}$ be Σ with the orientation reversed. Then $\overline{\Sigma}$ is a Seifert surface for $\overline{K}$ with orientation preserving smooth bicollar

$$\begin{aligned} \overline{\beta}\colon \overline{\Sigma} \times [-1,1] &\to S^3 \\ (x,t) &\mapsto \beta(x,-t) \end{aligned}$$

For all $i, j \in \{1, \dots, n\}$

$$\mathrm{lk}(e_i, \overline{\beta}(e_j \times \{1\}) = \mathrm{lk}(e_i, \beta(e_j \times \{-1\})) \overset{\text{Lemma 1.2}}{=} \mathrm{lk}(\beta(e_j \times \{-1\}), e_i) \overset{(*)}{=} \mathrm{lk}(e_j, \beta(e_i \times \{1\})$$

i.e. the Seifert matrix for $\overline{K}$ resulting from $\overline{\Sigma}$ and the basis $\{e_1, \dots, e_n\} \subseteq \mathrm{H}_1(\overline{\Sigma})$ is M^T.

- Let $\varphi\colon S^3 \to S^3$ be an orientation reversing diffeomorphism. Then $\Sigma^* := \varphi(\Sigma)$ is a Seifert surface for $K^* = \varphi(K)$ with an orientation preserving smooth bicollar

$$\begin{aligned} \beta^*\colon \Sigma^* \times [-1,1] &\to S^3 \\ (x,t) &\mapsto (\varphi \circ \beta)(\varphi^{-1}(x), -t) \end{aligned}$$

and $\{\varphi(e_1), \dots, \varphi(e_n)\} \subseteq \mathrm{H}_1(\Sigma^*)$ is a basis. Then for $i, j \in \{1, \dots, n\}$

$$\begin{aligned} \mathrm{lk}(\varphi(e_i), \beta^*(\varphi(e_j) \times \{1\})) &= \mathrm{lk}(\varphi(e_i), (\varphi \circ \beta)(e_j \times \{-1\})) \overset{\text{Lemma 1.2}}{=} -\mathrm{lk}(\beta(e_j \times \{-1\}), e_i) \\ &\underset{(*)}{=} -\mathrm{lk}(e_j, \beta(e_i \times \{1\})) \end{aligned}$$

i.e. the Seifert matrix for K^* resulting from Σ^* and the basis $\{\varphi(e_1), \dots, \varphi(e_n)\} \subseteq \mathrm{H}_1(\Sigma^*)$ is $-M^T$. ∎

1.3 Alexander polynomial

In principle, we have now defined a new knot invariant: The S-equivalence class of a Seifert matrix for the knot. Unfortunately, there is no apparent way to tell if two matrices are S-equivalent. For that, we need another invariant, this time one of S-equivalence classes – which will then also be a knot invariant. This is given by the Alexander polynomial:

Definition. Let K be an oriented knot or link and choose a Seifert matrix M for K. An *Alexander polynomial* of K is given by

$$\Delta_K(t) \coloneqq \det\Big(M - t \cdot M^T\Big) \in \mathbb{Z}[t]$$

Theorem 1.18. *The Alexander polynomial is up to multiplication by $\pm t^{\pm 1}$ a well-defined invariant of oriented links.*

Proof. Let K be an oriented knot or link with an $n \times n$ Seifert matrix M. By Lemma 1.16 it suffices to prove that elementary enlargements and congruences over $\mathbb{Z}$ only change the Alexander polynomial up to multiplication with $\pm t^{\pm 1}$:

- By Laplace expansion of the determinant

$$\det\left(\begin{pmatrix} M & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - t \cdot \begin{pmatrix} M & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^T\right) = \det\begin{pmatrix} M - t \cdot M^T & * & 0 \\ * & 0 & 1 \\ 0 & -t & 0 \end{pmatrix} = t \cdot \det(M - t \cdot M^T)$$

$$\det\left(\begin{pmatrix} M & 0 & 0 \\ * & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - t \cdot \begin{pmatrix} M & 0 & 0 \\ * & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^T\right) = \det\begin{pmatrix} M - t \cdot M^T & * & 0 \\ * & 0 & -t \\ 0 & 1 & 0 \end{pmatrix} = t \cdot \det(M - t \cdot M^T)$$

- We have $\det(P) = \det(P^T) \in \{\pm 1\}$ for any invertible $n \times n$ integer matrix P, therefore

$$\det\Big((P^T A P) - t \cdot (P^T A P)^T\Big) = \det(P^T) \cdot \det(A - t \cdot A^T) \cdot \det(P) = \det(A - t \cdot A^T) \blacksquare$$

In light of Theorem 1.18 we introduce the following notation:

Notation. Let $f, g \in \mathbb{Z}[t, t^{-1}]$ be Laurent polynomials. We write $f \doteq g$ if $f = \pm t^{\pm n} \cdot g$ for some $n \in \mathbb{N}_0$.

Example 1.19. We have already seen in Example 1.10 that a Seifert matrix for the n-component unlink L_n is given by an $(n-1) \times (n-1)$ zero matrix. Therefore is its Alexander polynomial given by[2]

$$\Delta_{L_n}(t) \doteq \det(0_{n-1}) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{else} \end{cases}$$

This leads us to the following convention:

Convention 1.20. Let K be a knot or link. We say the Alexander polynomial of K is *trivial* if

$$\Delta_K(t) \doteq \begin{cases} 1, & \text{if } K \text{ is a knot} \\ 0, & \text{else} \end{cases}$$

We continue calculating Alexander polynomials from the Seifert matrices encountered previously:

[2] We admit that the $n = 1$ case is slightly pathological. By definition the determinant of the unit matrix is always 1. But the 0×0 zero matrix is also the 0×0 unit matrix and therefore must have determinant 1. If one does not want to trust in this, one could of course choose a Seifert surface for the unknot of higher genus – for example a torus with an open disk removed – and redo the calculation.

Example 1.21. In Example 1.11 we saw that the matrices

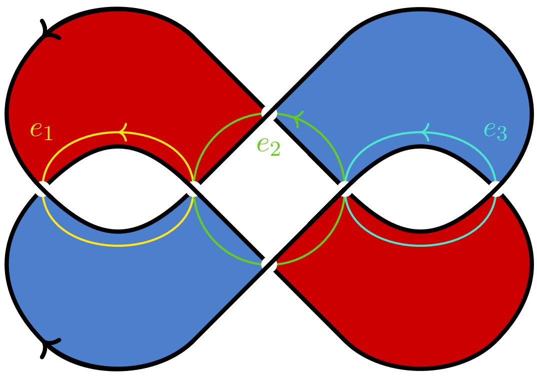
$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are Seifert matrices for the trefoil T. The Alexander polynomial of the trefoil is therefore given by

$$\Delta_T(t) \doteq t^2 - t + 1 \doteq t^3 - t^2 + t$$

Furthermore, we can now deliver on our promise from Example 1.3 to show that the Whitehead link is non-trivial:

Example 1.22. The Whitehead link W has a Seifert surface Σ given by



As per usual we coloured the surface such that the red side faces upwards. The Seifert matrix relative to the basis $\{e_1, e_2, e_3\} \subseteq \mathrm{H}_1(\Sigma)$ is then given by

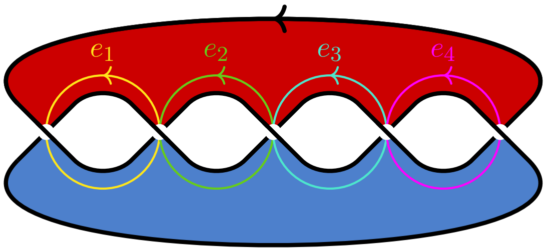
$$\begin{pmatrix} -1 & & \\ 1 & -1 & \\ & -1 & 1 \end{pmatrix}$$

It therefore has non-trivial Alexander polynomial

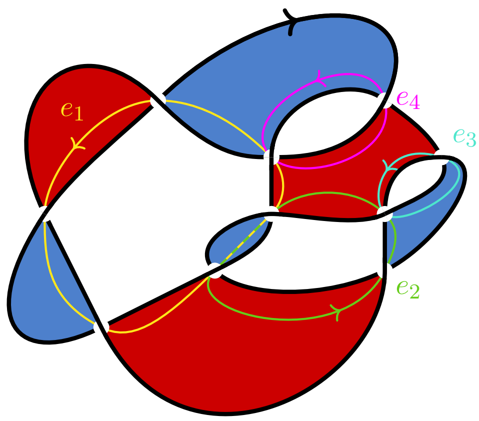
$$\Delta_W(t) \doteq -(t-1)^3 = -t^3 + 3t^2 - 3t + 1 \not\doteq 0$$

One might ask whether the Alexander polynomial is a complete knot invariant. The next example quickly dispels that hope:

Example 1.23. We consider the knots K_1 and K_2 indicated below and choose orientations such that the red side faces upwards.



K_1



K_2

The Seifert matrix M_1 (respectively M_2) for K_1 (respectively K_2) relative to $\{e_1, \dots, e_4\}$ is then given by

$$M_1 := \begin{pmatrix} -1 & & & \\ 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \end{pmatrix} \qquad M_2 := \begin{pmatrix} 3 & -1 & & \\ -1 & & & \\ & -1 & -1 & \\ -1 & & & 1 \end{pmatrix}$$

Therefore the Alexander polynomials of K_1 and K_2 are the same:

$$\Delta_{K_1}(t) \doteq \Delta_{K_2}(t) \doteq t^4 - t^3 + t^2 - t + 1$$

Remark. In the last example we saw that the Alexander polynomial cannot distinguish certain knots. But we motivated the Alexander polynomial by viewing it as an invariant for the S-equivalence class of a Seifert matrix. This begs the question, whether it is a complete invariant of S-equivalence. Unfortunately, it is not: The two Seifert matrices M_1 and M_2 are not S-equivalent, despite giving the same Alexander polynomial.
For any matrix M the matrix $M + M^T$ is symmetric. The signature

$$\sigma(M + M^T) := \#\{\text{positive eigenvalues of } M + M^T\} - \#\{\text{negative eigenvalues of } M + M^T\}$$

of this matrix is also invariant under S-equivalence of M.[3] Calculating this for M_1 and M_2 yields

$$\sigma\left(M_1 + M_1^T\right) = -4 \qquad \text{and} \qquad \sigma\left(M_2 + M_2^T\right) = 0$$

Hence M_1 and M_2 are not S-equivalent.

We have seen that the Alexander polynomial is not a complete invariant of knots. But one could still ask if it can detect the unknot. We will provide a class of knots with trivial Alexander polynomial only in Corollary 2.17. It is much easier to do this for links:

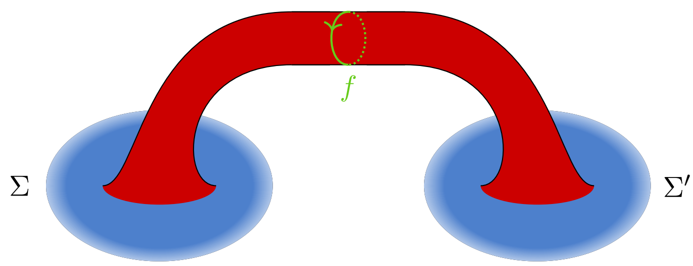
Proposition 1.24. *Let $L, L' \subseteq S^3$ be disjoint oriented knots or links. Assume there exists $\Sigma \subseteq S^3$ (respective $\Sigma' \subseteq S^3$) Seifert surface for L (respective L') such that $\Sigma \cap \Sigma' = \varnothing$. Then the Alexander polynomial of the link $L \cup L'$ is trivial:*

$$\Delta_{L \cup L'}(t) = 0$$

Proof. Let $P \in \Sigma \setminus \partial\Sigma$ and $Q \in \Sigma' \setminus \partial\Sigma'$. We chose a smooth embedding $\gamma\colon [0,1] \to S^3$ such that $\gamma(0) = P, \gamma(1) = Q$, $\gamma((0,1)) \cap (\Sigma \cup \Sigma') = \varnothing$ and

- a positive basis of $T_P\Sigma$ and $\gamma'(0)$ form a positive basis of S^3
- a positive basis of $T_Q\Sigma'$ and $\gamma'(1)$ form a negative basis of S^3.

Stabilization of $\Sigma \cup \Sigma'$ along γ produces a Seifert surface Λ for $L \cup L'$. Choose bases $\{e_1, \dots, e_n\} \subseteq \mathrm{H}_1(\Sigma)$ and $\{e'_1, \dots, e'_{n'}\} \subseteq \mathrm{H}_1(\Sigma')$. Let f be the cycle in the stabilization indicated below.



By a Mayer–Vietoris argument $\{e_1, \dots, e_n, e'_1, \dots, e'_{n'}, f\} \subseteq \mathrm{H}_1(\Lambda)$ is a basis. Let M be the Seifert matrix of $L \cup L'$ relative to this basis. The cycle f bounds a disc inside the stabilization, therefore the last row and column of M are equal to 0 by Lemma 1.7. Hence

$$\Delta_{L \cup L'}(t) \doteq \det(M - t \cdot M^T) = \det \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} = 0 \qquad \blacksquare$$

[3]Therefore this also gives a knot invariant, called the *signature of a knot*, see [Mur08, Chapter 6.4].

After all these examples we consider some basic properties of the Alexander polynomial:

Proposition 1.25. *Let $K \subseteq S^3$ be an oriented knot or link. The Alexander polynomial of K is symmetric in t and t^{-1}:*

$$\Delta_K(t) \doteq \Delta_K(t^{-1})$$

Furthermore, the mirror and reversion of K have the same Alexander polynomial as K:

$$\Delta_K(t) \doteq \Delta_{K^*}(t) \doteq \Delta_{\overline{K}}(t)$$

Proof. Let M be a Seifert matrix for K. Then

$$\Delta_K(t) \doteq \det\Big(M - t \cdot M^T\Big) \doteq \det\Big(-t^{-1} \cdot M + M^T\Big) = \det\Big(M - t^{-1} \cdot M^T\Big) \doteq \Delta_K(t^{-1})$$

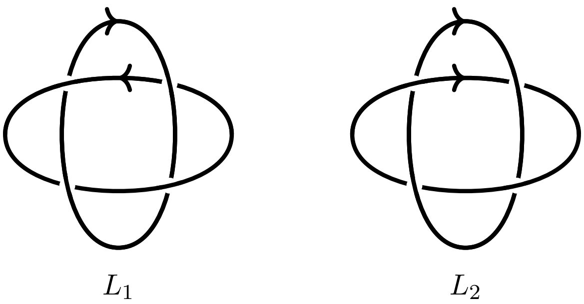
By Proposition 1.17 $-M$ is a Seifert matrix for K^*. Therefore is

$$\Delta_{K^*}(t) \doteq \det\Big((-M) - t \cdot (-M)^T\Big) \doteq \det\Big(M - t \cdot M^T\Big) \doteq \Delta_K(t)$$

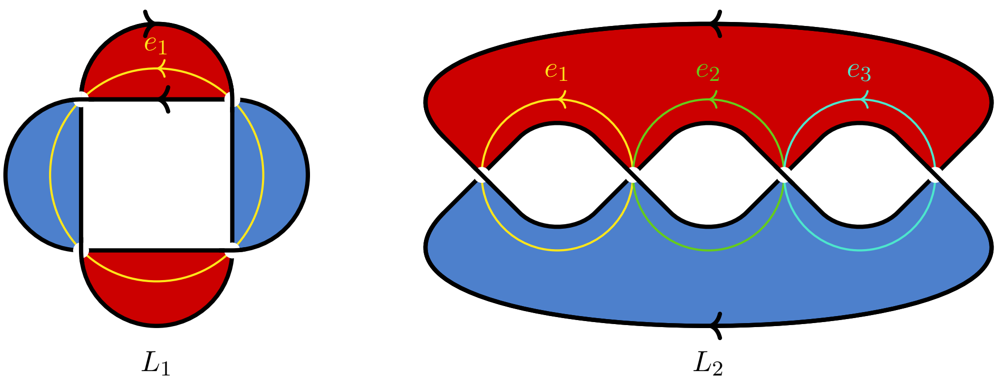
By Proposition 1.17 M^T is a Seifert matrix for $\overline{K}$. Therefore is

$$\Delta_{\overline{K}}(t) \doteq \det\Big(M^T - t \cdot (M^T)^T\Big) = \det\Big(\big(M - t \cdot M^T\big)^T\Big) = \det\Big(M - t \cdot M^T\Big) \doteq \Delta_K(t) \qquad \blacksquare$$

Remark. By Proposition 1.25 the Alexander polynomial does not depend on the orientation of a *knot*, i.e. it is an invariant of *knots without orientation*. This is different for *links*, as we can reverse the orientation of the components separately and thereby obtain different Alexander polynomials:
For example the links



differ only by reversing one of the orientations, but this changes their Seifert surface dramatically:



Calculating the Alexander polynomials from these yields

$$\Delta_{L_1}(t) \doteq -2t + 2 \qquad \text{and} \qquad \Delta_{L_2}(t) \doteq -t^3 + t^2 - t + 1$$

Hence we see that the orientation is important for the Alexander polynomial of a *link*.

We have already seen in a few places that the Alexander polynomial behaves differently with knots and links. In fact, it can distinguish between knots and links:

Proposition 1.26. *Let K be an oriented knot or link. Then*

$$\Delta_K(1) = \begin{cases} \pm 1, & \textit{if } K \textit{ is a knot} \\ 0, & \textit{else} \end{cases}$$

Proof. Let $\Sigma \subseteq S^3$ be an oriented Seifert surface for K with an orientation preserving smooth bicollar $\beta\colon \Sigma \times [0,1] \to S^3$.
We begin with a preliminary claim:

Claim. *Let $a, b \subseteq \Sigma$ be oriented knots that intersect transversally. Then* $\operatorname{lk}(a, b^+) - \operatorname{lk}(b, a^+)$ *does just depend on the abstract manifold Σ, not on the embedding $\Sigma \subseteq S^3$.*

Proof. Let $\{P_1, \dots, P_k\} := a \cap b \subseteq \Sigma$ be the intersection points of a and b. We consider the smooth homotopy

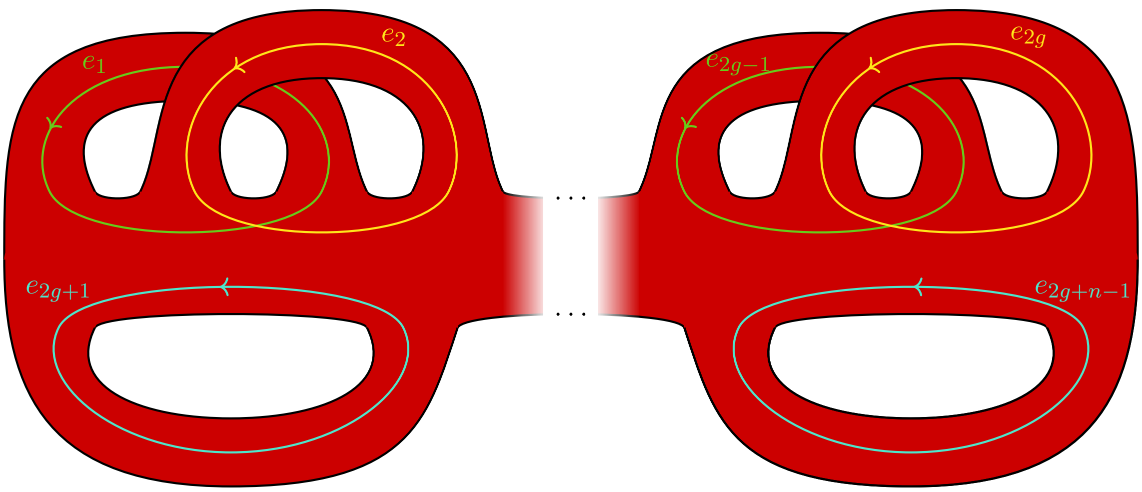
$$\begin{aligned} (a \sqcup b) \times [0,1] &\to S^3 \\ (x,t) &\mapsto \begin{cases} \beta(x,t), & \text{if } x \in a \\ (x, 1-t), & \text{if } x \in b \end{cases} \end{aligned}$$

between $a \cup b^+$ and $b^+ \cup a$. It is a smooth isotopy everywhere except in $\{P_1, \dots, P_k\}$. In these points it induces a crossing change. These change the linking number by

$$\begin{cases} +1, & \text{if the tangent vectors of } a \text{ and } b \text{ in } P_i \text{ form a positive basis of } T_{P_i}\Sigma \\ -1, & \text{if the tangent vectors of } a \text{ and } b \text{ in } P_i \text{ form a negative basis of } T_{P_i}\Sigma \end{cases},$$

which implies the claim. ☐

By the classification of surfaces there exists an orientation preserving diffeomorphism φ from an abstract surface of some genus g with n boundary components to Σ, where n is the number of components of L. We choose the basis for $\mathrm{H}_1(\Sigma)$ that is induced under this diffeomorphism by the basis indicated below:



abstract surface of genus g with n boundary components
oriented such that the red side faces upwards

Let $M = (\mathrm{lk}(\varphi(e_i), \varphi(e_j)^+))_{1\leq i,j\leq 2g+n-1}$ be the Seifert matrix of K relative to this basis. By the Claim

$$\Delta_K(1) = \det\Big(M - M^T\Big) = \det\Big(\big(\mathrm{lk}(\varphi(e_i), \varphi(e_j)^+) - \mathrm{lk}(\varphi(e_j), \varphi(e_i)^+)\big)_{1\leq i,j\leq 2g+n-1}\Big)$$

$$= \det\Big(\big(\mathrm{lk}(e_i, e_j^+) - \mathrm{lk}(e_j, e_i^+)\big)_{1\leq i,j\leq 2g+n-1}\Big)$$

$$= \det\begin{pmatrix} \underbrace{\begin{matrix} & 1 & & & \\ -1 & & & & \\ & & \ddots & & \\ & & & & 1 \\ & & & -1 & \end{matrix}}_{2g\times 2g} & \\ & 0_{n-1} \end{pmatrix} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{else} \end{cases}$$

This shows the claim for any Alexander polynomial of K by Theorem 1.18, since $\Delta_K(1)$ is the sum of the coefficients, i.e. multiplication with $\pm t^{\pm 1}$ only changes it up to a sign. ∎

It does not make sense to consider the degree of the Alexander polynomial since it is only defined up to multiplication by $\pm t^{\pm 1}$. Instead, we introduce the following notion:

Definition. Let $f = \sum_{n\in\mathbb{Z}} \lambda_n t^n \in \mathbb{Z}[t, t^{-1}]$ be a non-zero Laurent polynomial. The *breadth of f* is defined as

$$\mathrm{br}(f) := \max\{n \in \mathbb{Z} \mid \lambda_n \neq 0\} - \min\{n \in \mathbb{Z} \mid \lambda_n \neq 0\} \in \mathbb{N}_0$$

if $f \neq 0$ and $\mathrm{br}(0) := -\infty$.

It is clear that this is invariant under multiplication by $\pm t^{\pm 1}$, i.e. well-defined for the Alexander polynomial. It gives a lower bound on the genus of a knot:

Proposition 1.27. *Let L be an oriented n-component link of genus g. Then*

$$\mathrm{br}(\Delta_L(t)) \leq 2g + n - 1$$

Proof. Let Σ be a Seifert surface for L of genus g, i.e. $\mathrm{H}_1(\Sigma)$ is free on $2g + n - 1$ generators. Therefore many Seifert matrix for L relative to Σ is a $(2g + n - 1) \times (2g + n - 1)$ matrix and the degree of the Alexander polynomial $\det(M - t \cdot M^T) \in \mathbb{Z}[t]$ at most $2g + n - 1$. This gives an upper bound for the breadth of the Alexander polynomial, since the breadth of a polynomial is less than or equal to its degree. ∎

Remark. This formula allows us sometimes to determine the genus of a knot or link. For example, it can be used to show that the knots from Example 1.23 have genus 2. Unfortunately, we will later see in Corollary 2.17 that this bound can become arbitrarily bad.

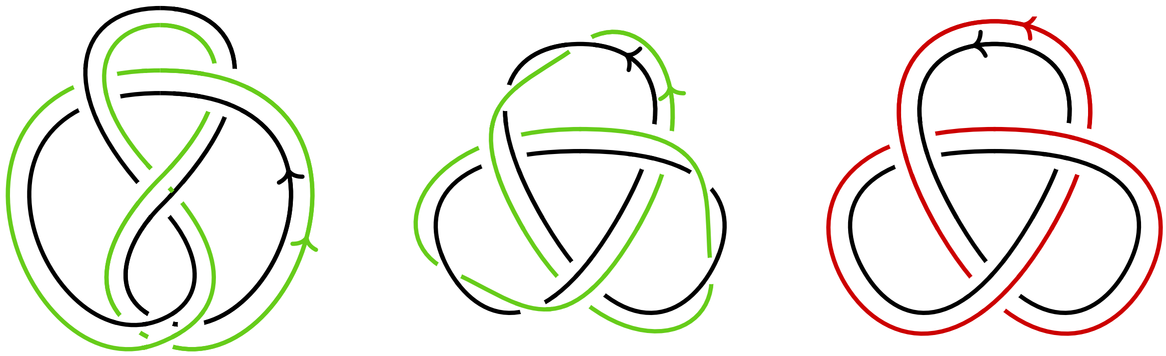
2 Satellite knots

2.1 Construction of satellite knots

In the second chapter, we will define satellite knots and give a formula for their Alexander polynomial. We begin by introducing some terminology:

Definition. Let $K \subseteq S^3$ be an oriented knot. A *thickening of* K is a smooth orientation-preserving embedding $\tau\colon S^1 \times \overline{B}^2 \to S^3$ that restricts to an orientation preserving diffeomorphism $S^1 \times \{0\} \to K$.[4] For any $x \in S^1$ the oriented submanifold $\tau(\{x\} \times S^1)$ is called a *meridian of* K.[5]
The thickening τ is called a *standard thickening* if for some $y \in S^1$ there exists a Seifert surface Σ of K such that $\tau(S^1 \times S^1) \cap \Sigma = \tau(S^1 \times \{y\})$. In this case the oriented submanifold $\tau(S^1 \times \{y\})$ is called a *longitude of* K.

Example 2.1.



longitude of the figure-8 knot — longitude of the trefoil — *not* a longitude of the trefoil

This shows that one has to be somewhat careful when drawing longitudes in a knot diagram, as they do in general *not* run parallel to the knot *in a diagram*. This is the case since by Lemma 1.4 the linking number of the knot and a longitude must be zero.

With this terminology in place, we can introduce the main character of this chapter:

Definition. We refer to a non-empty closed (oriented) 1-dimensional smooth submanifold of $S^1 \times \overline{B}^2$ as an *(oriented) pattern*. We call two (oriented) patterns $P, Q \subseteq S^1 \times \overline{B}^2$ *equivalent*, if there exists a smooth isotopy $F\colon P \times [0,1] \to S^1 \times \overline{B}^2$ with $P = \mathrm{im}(F_0)$ and $Q = \mathrm{im}(F_1)$ (such that $F_1 \circ F_0^{-1}\colon P \to Q$ is orientation preserving).
We call a pattern $P \subseteq S^1 \times \overline{B}^2$ *essential* if $\mathrm{im}(\varphi) \cap P \neq \varnothing$ for every smooth embedding $\varphi\colon \overline{B}^2 \to S^1 \times \overline{B}^2$ with $\varphi(S^1) \in \mathrm{H}_1(S^1 \times S^1)$ non-trivial.

Definition. Let $K \subseteq S^3$ be an oriented knot with a standard thickening $\tau\colon S^1 \times \overline{B}^2 \to S^3$ and $P \subseteq S^1 \times \overline{B}^2$ be an (oriented) pattern. Then $\tau(P)$ is called a *satellite link of* K *with pattern* P. It is called a *satellite knot* if P is connected. The knot K is the *companion* of the satellite $\tau(P)$.

[4] A thickening always exists by the Tubular Neighbourhood Theorem.

[5] We equip $\overline{B}^2 \subseteq \mathbb{C}$ with the canonical orientation, $S^1 = \partial\overline{B}^2$ with the boundary orientation and $S^1 \times \overline{B}^2$ with the product orientation. Then we consider the resulting orientation on $\tau(\{x\} \times S^1)$

Remark. The condition for a standard thickening ensures that the solid torus is not getting twisted itself in the embedding. This is important if we want the satellite to be fully described by the original knot and the pattern. The restriction we give appears somewhat artificial at the moment. We will later see in Proposition 2.15 why it achieves the desired outcome.

The next lemma states that satellites are unique in a suitable way. As we will not use it in our proof of the formula for the Alexander polynomial of satellites, we only sketch a proof.[6]

Lemma 2.2. *Let $K, K' \subseteq S^3$ be equivalent oriented knots and let τ (respectively τ') be standard thickenings of K (respectively K'). Let furthermore $P, P' \subseteq S^1 \times \overline{B}^2$ be equivalent (oriented) patterns. Then the (oriented) knots or links $\tau(P)$ and $\tau'(P')$ are equivalent.*

Sketch of proof. By the smooth Isotopy Extension Theorem (see [Fri23, Isotopy Extension Theorem]) there exists an diffeotopy $G\colon S^3 \times [0,1] \to S^3$ such that $G_0 = \mathrm{id}$, $G_1(K) = K'$ and $G_1 \circ G_0^{-1}\colon K \to K'$ is orientation preserving. By the Tubular Neighbourhood Theorem (see [Wal16, Theorem 2.5.5]) and since the thickenings are standard and the orientations of K and K' are preserved by G, we can additionally assume that $G_1 \circ \tau = \tau'$. This implies that $G_1(\tau(P)) = \tau'(P)$, in particular, the links $\tau(P)$ and $\tau'(P)$ are equivalent. The equivalence of the patterns P and P' gives the remaining part of the isotopy between the oriented knots $\tau(P)$ and $\tau(P')$. ■

Remark. In Lemma 2.2 we require the knots K and K' to be equivalent as *oriented* knots. This is needed to ensure that the standard thickening travels around both knots in the same direction. If the knots were just equivalent as knots *without orientation* the situation would not be as clear. In particular, one could easily reverse the orientation of the knot by a suitable choice of oriented pattern. This is the reason we only consider satellites of oriented knots.

A converse of Lemma 2.2 holds for checking if a satellite is the unknot:

Proposition 2.3. *Let $K \subseteq S^3$ be a non-trivial oriented knot and $P \subseteq S^1 \times \overline{B}^2$ be a connected essential pattern. Then the satellite of K with pattern P is not the unknot.*

Proof. see [Rol03, Theorem 4.D.9]. ■

On some occasions, we want to view a pattern as a knot in its own right. For this, we introduce the following convention:

Convention 2.4. We refer to the smooth embedding

$$\begin{aligned} \Theta\colon S^1 \times \overline{B}^2 &\to S^3 \subseteq \mathbb{C}^2 \\ (x, y) &\mapsto \frac{(x, y)}{\|(x, y)\|} \end{aligned}$$

as the *standard embedding of the solid torus into* S^3. It is a standard thickening for the unknot $\{(x, 0) \in S^3 \subseteq \mathbb{C}^2\}$.
We view a (oriented) pattern $P \subseteq S^1 \times \overline{B}^2$ as an (oriented) knot or link in S^3 by considering it as a satellite of an unknot, for example $\Theta(P)$. Lemma 2.2 ensures that this determines the knot or link up to equivalence.

[6] To be quite honest, our condition on a standard thickening is entirely unsuitable for proving this.

2.2 Examples of satellite knots

2.2.1 Cable knots

To find interesting examples of satellite knots, we have to find nice knots inside a solid torus. We begin with the knots that are contained in its boundary:

Definition. Let $p, q \in \mathbb{Z}$ not both zero. We consider the smooth embedding of the torus[7] $\mathbb{R}^2/\mathbb{Z}^2$ into a solid torus given by

$$\begin{aligned} \Phi\colon \mathbb{R}^2/\mathbb{Z}^2 &\to S^1 \times \overline{B}^2 \\ (x, y) &\mapsto (\exp(2\pi\,\mathrm{i}\,x), \exp(2\pi\,\mathrm{i}\,y)) \end{aligned}$$

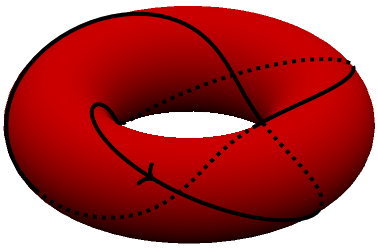
The image of the set

$$X := \{(x, y) \in \mathbb{R}^2 \mid qx - py \in \mathbb{Z}\} \subseteq \mathbb{R}^2$$

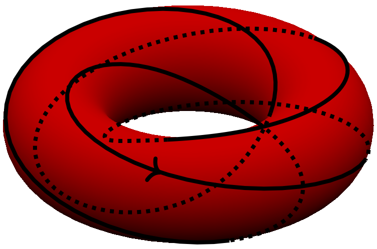
under Φ is called the (p, q)-*torus link* $\mathrm{T}(p, q) := \Phi(X)$. If it has one component, we refer to it as a *torus knot*.[8]

The *standard orientation for* $\mathrm{T}(p, q)$ is obtained from orienting the submanifold $X \subseteq \mathbb{R}^2$ by choosing (q, p) as a positive basis in every tangent space.

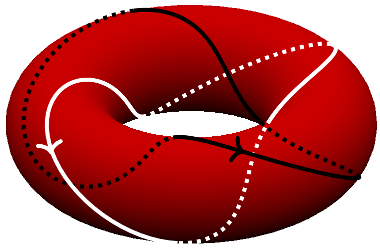
Example 2.5.



(2, 3)-torus knot



(4, 3)-torus knot

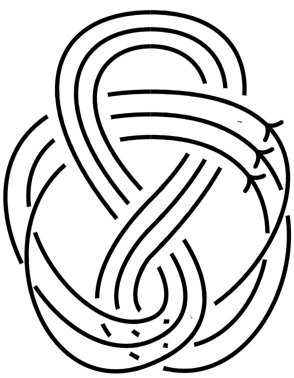


(2, 4)-torus link

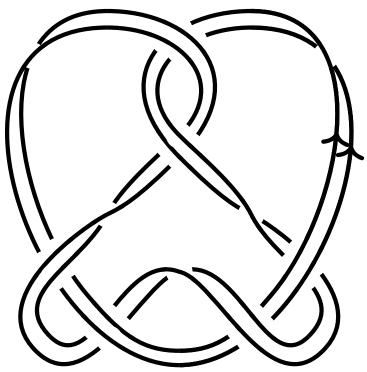
Since we defined the torus knots as subsets of a solid torus and then transported them to S^3 instead of directly defining them as submanifolds of S^3, the reader might have already guessed that we want to use them as a pattern:

Definition. Let $K \subseteq S^3$ be a knot and $p, q \in \mathbb{Z}$ not both zero. The satellite of K with pattern $\mathrm{T}(p, q)$ is called the (p, q)-*cable of* K.

Example 2.6.



(3,2)-cable of the figure-8 knot



(2,1)-cable of Stevedore's knot

[7] The group action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ is the usual one given by component-wise addition.

[8] We use Convention 2.4 to view subsets of a solid torus $S^1 \times \overline{B}^2$ as links in S^3.

2.2.2 Connected sum of knots

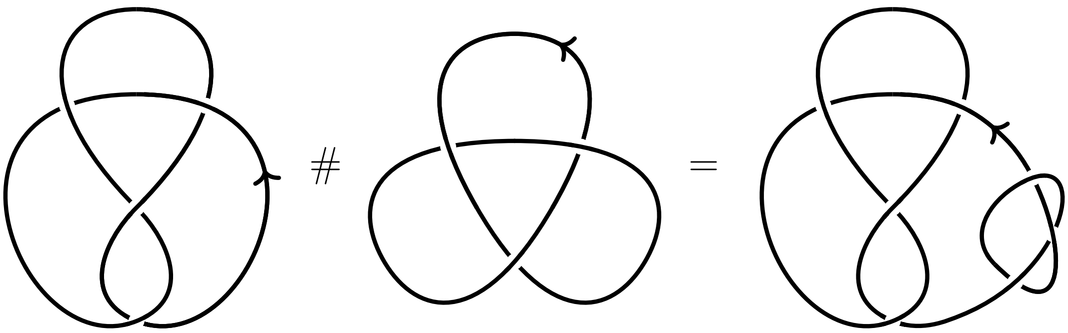
A further class of examples for satellites is the connected sum of knots. We use the satellite operation to define the connected sum. This is not the standard approach to do so – It is not difficult to prove that the result is equivalent to the definition from [Lic97, p.4] or [Rol03, p.40].

Definition. Let $J, K \subseteq S^3$ be oriented knots. We choose a smooth orientation-preserving embedding $\varphi\colon S^1 \times \overline{B}^2 \to S^3$ such that

- $K \subseteq \operatorname{im}(\varphi)$
- $K \cap \varphi\Big(\{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \overline{B}^2\Big) = \varphi(\{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \{0\})$
- $\varphi\colon \{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \{0\} \to K$ is orientation preserving

The satellite of J with pattern $\varphi^{-1}(K)$ is called the *connected sum $J\#K$ of J and K.*

Example 2.7.



As the reader expects, this definition does not depend on any choices. Again we will not use this and therefore only sketch a proof:

Lemma 2.8. *Let $J_1, J_2, K_1, K_2 \subseteq S^3$ be oriented knots where J_1 and J_2 (respectively K_1 and K_2) are equivalent. For any smooth orientation-preserving embeddings $\varphi_1, \varphi_2\colon S^1 \times \overline{B}^2 \to S^3$ such that for $i = 1, 2$*

- $K_i \subseteq \operatorname{im}(\varphi_i)$
- $K_i \cap \varphi_i\Big(\{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \overline{B}^2\Big) = \varphi_i(\{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \{0\})$
- $\varphi\colon \{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \{0\} \to K$ *is orientation preserving*

the satellite of J_1 with pattern $\varphi_1^{-1}(K_1)$ and the satellite of J_2 with pattern $\varphi_2^{-1}(K_1)$ are equivalent.

Sketch of proof. We may assume that there exists an interval on which K_1 and K_2 agree, including their orientations. The complement of a small enough thickening of a meridian of K_1 and K_2 around that interval is an embedded solid torus $T \subseteq S^3 \setminus (K_1 \cup K_2)$. An appropriate choice of embedding $\varphi\colon S^1 \times \overline{B}^2 \to T \subseteq S^3$ fulfils the properties from the definition for K_1 and K_2. Since K_1 and K_2 are equivalent, we deduce that $\varphi^{-1}(K_1), \varphi^{-1}(K_2) \subseteq S^1 \times \overline{B}^2$ are equivalent patterns, hence by Lemma 2.2 the satellite of J_1 with pattern $\varphi^{-1}(K_1)$ and the satellite of J_2 with pattern $\varphi^{-1}(K_1)$ are equivalent knots or links.
Any two embeddings constructed in this manner are smoothly isotopic up to twisting the solid torus. These twist can be pushed into $\varphi\Big(\{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \overline{B}^2\Big)$ hence they do not affect $\varphi^{-1}(K)$. This proves the claim since any embedding fulfilling the given properties can be smoothly isotoped to be of the described form. ∎

The experienced reader might miss the statement that the connected sum of knots is commutative. The given definition is unfortunately not really suited for proving this. One could probably force out a proof, but to be honest one is better off proving equivalence to a more standard definition – where commutativity should be apparent. But in contrast to such a definition, we can quite easily prove non-triviality:

Proposition 2.9. *Let $J, K \subseteq S^3$ be oriented knots. The connected sum $J\#K$ of J and K is trivial if and only if both J and K are trivial.*

Proof. It is clear that the sum of trivial knots is trivial.
Assume J and K are non-trivial. We choose a orientation-preserving smooth embedding $\varphi\colon S^1 \times \overline{B}^2 \to S^3$ such that

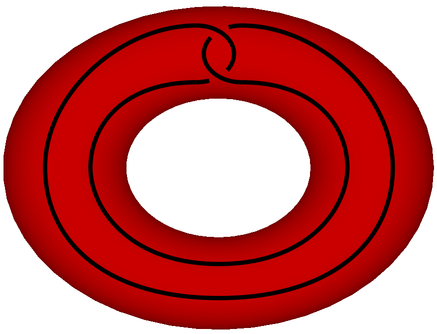
- $K \subseteq \operatorname{im}(\varphi)$
- $K \cap \varphi\Big(\{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \overline{B}^2\Big) = \varphi(\{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \{0\})$
- $\varphi\colon \{a + b\,\mathrm{i} \in S^1 \mid a \leq 0\} \times \{0\} \to K$ is orientation preserving

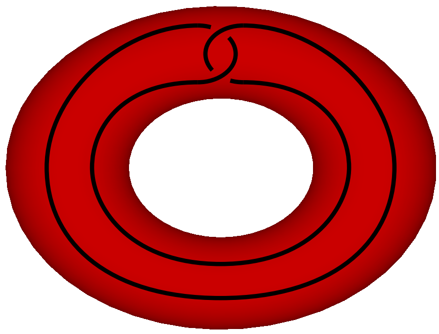
By Proposition 2.3 suffices it to show that $\varphi^{-1}(K) \subseteq S^1 \times \overline{B}^2$ is an essential pattern:
Suppose it is not. Then there exists a smooth embedding $\psi\colon \overline{B}^2 \to S^1 \times \overline{B}^2$ with $\psi(S^1) \in \mathrm{H}_1(S^1 \times \overline{S}^1)$ non-trivial such that $\operatorname{im}(\psi) \cap \varphi^{-1}(K) = \varnothing$. By Lemma 1.4 $\operatorname{lk}(\psi(S^1), \varphi^{-1}(K)) = 0$. The second condition on φ ensures that $\operatorname{lk}(\{-1\} \times S^1, \varphi^{-1}(K)) = \pm 1$. By Lemma 1.4 these two linking numbers are equal since $\psi(S^1)$ and $\{-1\} \times S^1$ are homologous. Contradiction! ∎

2.2.3 Whitehead double of a knot

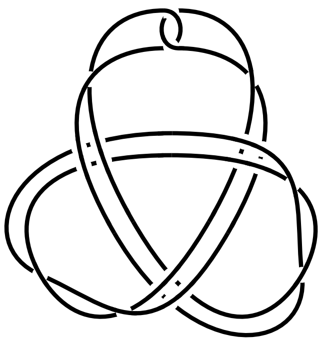
Our last class of examples is given by Whitehead doubles. It is an incongruous example, because we want to consider it for knots *without orientation*.

Definition. Let $K \subseteq S^3$ be a knot. Choose any orientation on K. A satellite knot of K with one of the patterns indicated below is called a *Whitehead double of K*.

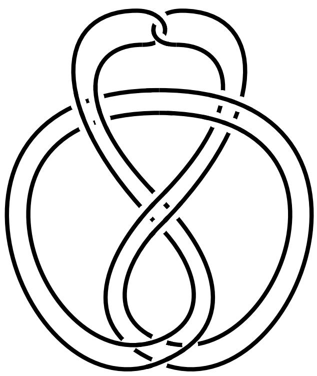




Example 2.10.



Whitehead double of the trefoil



Whitehead double of the figure-8 knot

Remark. In the definition of a Whitehead double we could have distinguished between the two given patterns. The resulting satellite knot would then depend on the orientation of the original knot. We have not done so, since the Alexander polynomial studied here cannot detect a reversion of orientation by Proposition 1.25.

Due to this remark, we defined the Whitehead double as an operation on knots without orientation. Therefore the uniqueness does not immediately follow from Lemma 2.2, but has to be stated separately:

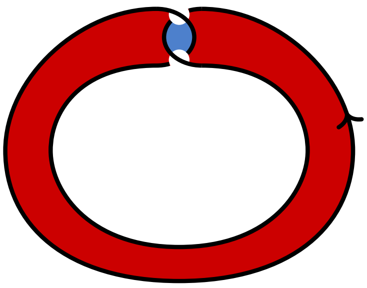
Lemma 2.11. *Let $K, K' \subseteq S^3$ be equivalent knots. Then a Whitehead doubles of K is equivalent to a Whitehead double of K'.*

Proof. By Lemma 2.2 we only need to consider the effect of the choice of orientation. The claim follows since witching orientations precisely corresponds to switching between the two given patterns. ■

The Whitehead double of a knot is very close to being unknotted: a single crossing change will make it trivial. Nevertheless, the following proposition holds:

Proposition 2.12. *The Whitehead double of a non-trivial knot has genus 1, in particular, it is non-trivial.*

Proof. There is a Seifert surface of genus 1 for the pattern P of a Whitehead double that is fully contained in the solid torus:



Seifert surface consisting of an annulus and a disc above it
with two twisted bands between them
Contrary to our usual convention both red and blue face upwards.

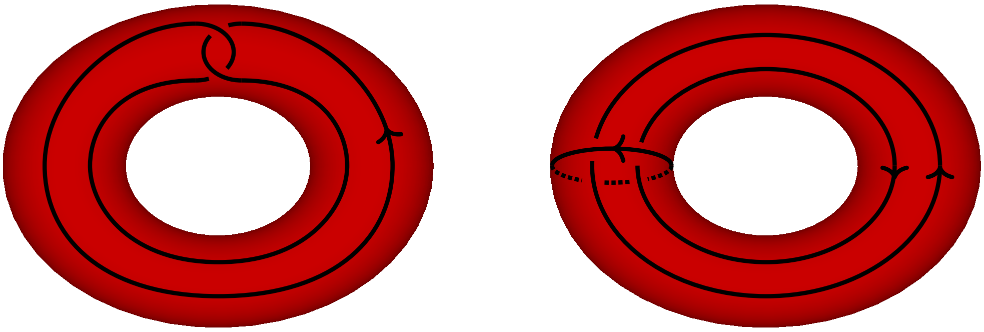
Pushing this through the standard thickening gives a genus 1 Seifert surface for any Whitehead double. Hence, we only need to prove that the Whitehead double of a non-trivial knot is non-trivial:
By Proposition 2.3 it suffices to prove that the pattern $P \subseteq S^1 \times \overline{B}^2$ used to define Whitehead doubles is essential. Suppose it is not, i.e. that there exists a smooth embedding $\varphi\colon \overline{B}^2 \to S^1 \times \overline{B}^2$ with $\varphi(S^1) \in \mathrm{H}_1(S^1 \times S^1)$ non-trivial such that $\mathrm{im}(\varphi) \cap P = \varnothing$. Then $P \cup \varphi(S^1)$ is the Whitehead link. We have seen in Example 1.22 that it has non-trivial Alexander polynomial.
But by a compactness argument around Theorem 1.5 we can show that there exists a Seifert surface $\Sigma \subseteq S^3$ for P such that $\Sigma \cup \varphi\left(\overline{B}^2\right) \subseteq S^3$ is a compact orientable 2-dimensional smooth submanifold with two path-components. This implies by Proposition 1.24 that the Alexander polynomial of $P \cup \varphi(S^1)$ is trivial. Contradiction! ■

2.3 Alexander polynomial of satellite knots

In this section, we want to give a formula for the Alexander polynomial of satellite knots. To this end, we need to introduce some further terminology:

Definition. Let $P \subseteq S^1 \times \overline{B}^2$ be an oriented pattern. The *winding number* of P is the unique natural number $n \in \mathbb{N}_0$ such that P represents n times a generator of $\mathrm{H}_1\left(S^1 \times \overline{B}^2\right) \cong \mathbb{Z}$.

Example 2.13. Two oriented patterns of winding number 0:



To calculate the Alexander polynomial of a satellite, we need to construct a Seifert surface for it:

Construction 2.14. Let $K \subseteq S^3$ be an oriented knot with a standard thickening τ and $P \subseteq S^1 \times \overline{B}^2$ be an oriented pattern with winding number n. We first construct a partial Seifert surface for P contained in $S^1 \times \overline{B}^2$: The solid torus $S^1 \times \overline{B}^2$ projects onto an annulus $S^1 \times [-1,1]$ embedded into $\mathbb{R}^2$. We may assume that this projection results in a diagram for P in $S^1 \times [-1,1]$. We can now proceed similarly to Seifert's algorithm:[9]
After removing all crossings in the usual orientation compatible way, we obtain a set of embedded circles in $S^1 \times [-1,1]$. These are either null-homotopic in $S^1 \times [-1,1]$ or a generator of $\pi_1(S^1 \times [-1,1])$. We cap off all circuits that are null-homotopic with disks. Next, we add annuli between any two adjacent circuits that go around $S^1 \times [-1,1]$ in opposite directions. As usual, we might have to stack these disks and annuli. In doing so, we make sure that they are still within $S^1 \times \overline{B}^2$. The remaining circles go around $S^1 \times \overline{B}^2$ in the same direction. For each of them we add a vertical annulus connecting it to an embedded circle in $S^1 \times S^1$. We can always do this, since we previously only capped off adjacent pairs of circles. Lastly, we add twisted bands at the crossings to obtain a compact oriented 2-dimensional smooth submanifold $\Gamma \subseteq S^1 \times \overline{B}^2$ whose oriented boundary is given by P and coherently oriented embedded circles $l_1, \ldots, l_n \subseteq S^1 \times S^1$. The number of these is n, since P is homologous to their union. If Γ is not connected, we stabilize appropriately to make it connected.
Since τ is a standard thickening, there exists some $y \in S^1$ such that there is a Seifert surface Σ for K with $l := \tau(S^1 \times \{y\}) = \tau(S^1 \times S^1) \cap \Sigma$. We can assume that this intersection is transversal (see [Wal16, Chapter 4.5]). By Theorem 1.6 there exists an orientation preserving smooth bicollar $\beta\colon \Sigma \times [-1,1] \to S^3$. After a smooth isotopy of the bicollar we can additionally assume that

$$\beta(l \times [0,1]) = \mathrm{im}(\beta) \cap \tau\left(S^1 \times S^1\right)$$

Let $\Sigma' := (\Sigma \setminus \mathrm{im}(\tau)) \cup l$. We may assume after a smooth isotopy in $S^1 \times \overline{B}^2$ that there exist $x_1 < \cdots < x_n \in [0,1)$ such that $l_i = \beta(l \times \{x_i\})$ for $i \in \{1, \ldots, n\}$ and such that

$$\Lambda := \tau(\Gamma) \cup \bigcup_{i=1}^{n} \beta\left(\Sigma' \times \{x_i\}\right)$$

is a Seifert surface for $\tau(P)$, i.e. the satellite of K with pattern P.

[9] We abuse the similarity to Seifert's algorithm to be a bit vague in the following paragraph. All issues occurring here also arise in a similar way in Seifert's algorithm. As we trust the experienced reader to know Seifert's algorithm, we will not give a detailed description of these slightly bothersome matters.

Definition. Let $K \subseteq S^3$ be an oriented knot with Seifert surface Σ and $P \subseteq S^1 \times \overline{B}^2$ be an oriented pattern. The Seifert surface Λ constructed in Construction 2.14 is the *satellite Seifert surface* for the satellite of K with pattern P resulting from Σ.

Before we come to the main theorem, we can now describe in what sense standard thickenings are untwisted. This will also turn out useful in proving the main theorem:

Proposition 2.15. *Let $K \subseteq S^3$ be an oriented knot with a standard thickening $\tau\colon S^1 \times \overline{B}^2 \to S^3$. For two disjoint oriented patterns $P, Q \subseteq S^1 \times \overline{B}^2$ we have*[10]

$$\operatorname{lk}(P, Q) = \operatorname{lk}(\tau(P), \tau(Q))$$

Proof. Let τ be a standard thickening of K. Let Λ be the satellite Seifert surface for $\tau(P)$, the satellite of K with pattern P. Let Λ_0 be the satellite Seifert surface for $\Theta(P)$, i.e. the satellite of the unknot with pattern P. We need compatible bicollars for Λ and Λ_0: Let $\beta_0\colon \Lambda_0 \times [-1,1] \to S^3$ be an orientation preserving smooth bicollar. We may assume that $\beta_0(\Theta(\Gamma) \times [-1,1]) \subseteq \operatorname{im}(\Theta)$. Hence, $\tau \circ \Theta^{-1} \circ \beta_0$ is an orientation preserving smooth bicollar of $\tau(\Gamma)$. It follows essentially from [Wal16, p.57] that we can extend this to an orientation preserving smooth bicollar of Λ. The claim now follows by calculating the linking numbers from these Seifert surfaces and bicollars using Lemma 1.7. ∎

Now we can state and prove the main theorem of this chapter:

Theorem 2.16. *Let $K \subseteq S^3$ be an oriented knot and $P \subseteq S^1 \times \overline{B}^2$ be an oriented pattern with winding number n. Let L be the satellite of K with pattern P. Then*

$$\Delta_L(t) \doteq \Delta_P(t) \cdot \Delta_K(t^n)$$

Proof. We continue in the notation from Construction 2.14, in particular, let

$$\Lambda \coloneqq \tau(\Gamma) \cup \bigcup_{i=1}^{n} \beta\left(\Sigma' \times \{x_i\}\right)$$

be the satellite Seifert surface for the satellite of K with pattern P.
Let $e_1, \dots, e_k \subseteq \Sigma$ be oriented knots such that $\{e_1, \dots, e_k\} \subseteq \mathrm{H}_1(\Sigma')$ is a basis. The satellite Seifert surface Λ_0 for P as a satellite of the unknot with standard thickening Θ consists of Γ and n discs. There exist smoothly embedded circles $f_1, \dots, f_m \subseteq \Gamma$ such that $\{f_1, \dots, f_m\} \subseteq \mathrm{H}_1(\Gamma)$ is a basis. It follows from a Mayer–Vietoris argument that

$$\{\tau(f_1), \dots, \tau(f_k)\} \cup \{\beta\left(e_i \times \{x_i\}\right) \mid x \in \{1, \dots, n\}\} \subseteq \mathrm{H}_1(\Delta)$$

is a basis.
Let $\beta'\colon \Lambda \times [-1,1] \to S^3$ be an orientation preserving smooth bicollar of Λ. We may assume that $\beta'(\tau(\Gamma) \times [-1,1]) \subseteq \operatorname{im}(\tau)$ and that $\operatorname{im}(\beta') \subseteq \operatorname{im}(\beta)$. Hence, $\tau \circ \Theta^{-1} \circ \beta$ is an orientation preserving smooth bicollar of $\Theta(\Gamma)$. If follows essentially from [Wal16, p.57] that we can extend this to an orientation preserving smooth bicollar of Λ_0. Let M be the Seifert matrix for P resulting from Λ_0 and this bicollar. By Proposition 2.15 is the Seifert matrix of $\tau(P)$ resulting from Λ is equal to

$$\begin{pmatrix} M & N \\ N' & X \end{pmatrix}$$

for appropriate matrices N, N' and X.
The satellite Seifert surfaces for $f_1, \dots, f_k$ and $f_1^+, \dots, f_k^+$ as satellites of K can be taken to be disjoint from any given knot in $\operatorname{im}(\beta) \setminus \operatorname{im}(\tau)$. Hence, we can show by calculating the linking

[10] As linking numbers are only defined in S^3 we make use of Convention 2.4 as usual.

numbers via Lemma 1.7 that N and N' are zero matrices. The matrix X is the $n \times n$ block matrix

$$X = \begin{pmatrix} A & A & A & \dots & A \\ A^T & A & A & \dots & A \\ A^T & A^T & A & \dots & A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^T & A^T & A^T & \dots & A \end{pmatrix}$$

where A is the Seifert matrix for K resulting from Σ and the basis $\{e_1, \dots, e_k\} \subseteq \mathrm{H}_1(\Sigma)$. This is the case since for all pairs of knots whose linking number appears in X there is an appropriate equivalent pair of knots whose linking number appears in A. It can be found by shifting the higher of the knots into $\beta(\Sigma' \times \{1\})$ and the lower into $\beta(\Sigma' \times \{0\})$. We therefore calculate that

$$\begin{aligned}
\det(X - t\cdot X^T) &= \det\left(\begin{pmatrix} A & A & A & \dots & A \\ A^T & A & A & \dots & A \\ A^T & A^T & A & \dots & A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^T & A^T & A^T & \dots & A \end{pmatrix} - t\cdot \begin{pmatrix} A & A & A & \dots & A \\ A^T & A & A & \dots & A \\ A^T & A^T & A & \dots & A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^T & A^T & A^T & \dots & A \end{pmatrix}^T\right) \\
&= \det\begin{pmatrix} A - t\cdot A^T & A - t\cdot A & A - t\cdot A & \dots & A - t\cdot A \\ A^T - t\cdot A^T & A - t\cdot A^T & A - t\cdot A & \dots & A - t\cdot A \\ A^T - t\cdot A^T & A^T - t\cdot A^T & A - t\cdot A^T & \dots & A - t\cdot A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^T - t\cdot A^T & A^T - t\cdot A^T & A^T - t\cdot A^T & \dots & A - t\cdot A^T \end{pmatrix} \\
&\overset{\substack{\text{replace first row by} \\ \sum_{i=0}^{n-1} t^i\cdot(\text{row } i+1)}}{\doteq} \det\begin{pmatrix} A - t^n\cdot A^T & A - t^n\cdot A^T & A - t^n\cdot A^T & \dots & A - t^n\cdot A^T \\ A^T - t\cdot A^T & A - t\cdot A^T & A - t\cdot A & \dots & A - t\cdot A \\ A^T - t\cdot A^T & A^T - t\cdot A^T & A - t\cdot A^T & \dots & A - t\cdot A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^T - t\cdot A^T & A^T - t\cdot A^T & A^T - t\cdot A^T & \dots & A - t\cdot A^T \end{pmatrix} \\
&\overset{\substack{\text{subtract first column} \\ \text{from other columns}}}{\doteq} \det\begin{pmatrix} A - t^n\cdot A^T & 0 & 0 & \dots & 0 \\ * & A - A^T & * & \dots & * \\ * & 0 & A - A^T & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \dots & A - A^T \end{pmatrix} \\
&\overset{\substack{\text{by Proposition 1.26} \\ \det(A - A^T) = \Delta_K(1) = \pm 1}}{\doteq} (\pm 1)^{n-1}\cdot \det\left(A - t^n\cdot A^T\right) \doteq \Delta_K(t^n)
\end{aligned}$$

Therefore is

$$\begin{aligned}
\Delta_L(t) &\doteq \det\left(\begin{pmatrix} M & \\ & X \end{pmatrix} - t\cdot \begin{pmatrix} M & \\ & X \end{pmatrix}\right) = \det\left(M - t\cdot M^T\right)\cdot \det\left(X - t\cdot X^T\right) \\
&\doteq \Delta_P(t)\cdot \Delta_K(t^n)
\end{aligned}$$
■

As a corollary we describe the Alexander polynomial for the connected sum of knots and for Whitehead doubles:

Corollary 2.17.

(i) *Let $J, K \subseteq S^3$ be oriented knots. The Alexander polynomial of the connected sum $J \# K$ is given by*

$$\Delta_{J\#K}(t) \doteq \Delta_J(t)\cdot \Delta_K(t)$$

(ii) *The Alexander polynomial of the Whitehead double of any knot is equal to 1.*

Proof.

(i) The second condition in the definition of the connected sum of knots ensures that the winding number is 1 in this case. The claim now follows from Theorem 2.16.

(ii) The pattern P for the Whitehead double W of a knot K has winding number 0. Therefore

$$\Delta_W(t) \underset{\substack{\uparrow \\ \text{Theorem 2.16}}}{\doteq} \Delta_P(t) \cdot \Delta_K(t^0) \underset{\substack{\uparrow \\ \text{Proposition 1.26}}}{\doteq} \Delta_P(t) = 1$$

since $\Theta(P) \subseteq S^3$ is the unknot. ∎

Remark. We can now see that the lower bound on the genus of a knot given by the breadth of its Alexander polynomial from Proposition 1.27 is not always sharp: By Proposition 2.12 the genus of the Whitehead double of a non-trivial knot is 1 and by Corollary 2.17 the breadth of its Alexander polynomial is 0.
In fact, this bound can become arbitrarily bad: Let W be the Whitehead double of any non-trivial knot. The g-fold connected sum of W with itself has genus g, since the genus of a knot is additive under connected sum (see [Lic97, Theorem 2.4]), but the breadth of its Alexander polynomial is still 0 by Corollary 2.17.

3 Torus knots

3.1 Basic properties of torus knots

In Section 2.2.1 we used torus knots to define cables of knots. At that point, we did not develop any theory about torus knots – a shortcoming that we want to remedy in this chapter. For convenience, we begin by recalling their definition:[11]

Definition. Let $p, q \in \mathbb{Z}$ not both zero. We consider the smooth embedding of the torus $\mathbb{R}^2/\mathbb{Z}^2$ into S^3 given by

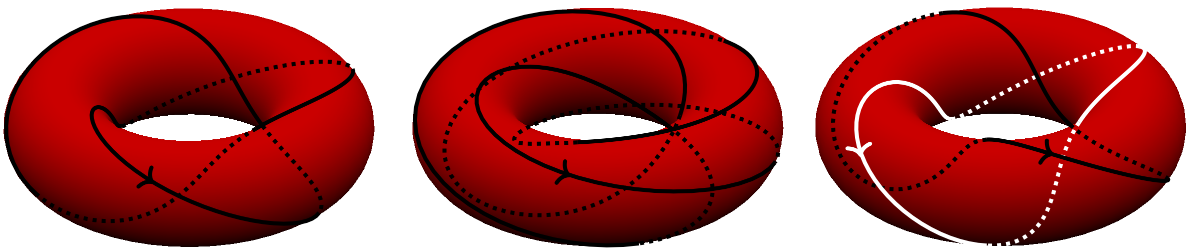
$$\begin{aligned} \Phi \colon \mathbb{R}^2/\mathbb{Z}^2 &\to \Theta\left(S^1 \times \overline{B}^2\right) \subseteq S^3 \\ (x, y) &\mapsto \frac{1}{\sqrt{2}}(\exp(2\pi\,\mathrm{i}\,x), \exp(2\pi\,\mathrm{i}\,y)) \end{aligned}$$

The image of the set

$$X := \{(x, y) \in \mathbb{R}^2 \mid qx - py \in \mathbb{Z}\} \subseteq \mathbb{R}^2$$

under Φ is called the (p, q)-*torus link* $\mathrm{T}(p, q) := \Phi(X)$. If it has one component, we refer to it as a *torus knot.*
The *standard orientation for* $\mathrm{T}(p, q)$ is obtained from orienting the submanifold $X \subseteq \mathbb{R}^2$ by choosing (q, p) as a positive basis in every tangent space.

Example 2.5.



$(2, 3)$-torus knot $\qquad$ $(4, 3)$-torus knot $\qquad$ $(2, 4)$-torus link

One might ask for a way to determine if a torus link is supposed to be called a torus knot. The next lemma gives even a bit more than that:

Lemma 3.1. *Let $p, q \in \mathbb{Z}$ not both zero. The link $\mathrm{T}(p, q)$ has $\gcd(p, q)$ components, in particular*

$$\mathrm{T}(p, q) \textit{ is a knot} \quad \Leftrightarrow \quad p, q \textit{ coprime}$$

Furthermore, each path-component of $\mathrm{T}(p, q)$ is equivalent to $\mathrm{T}(p/\gcd(p, q), q/\gcd(p, q))$.

Proof. We consider the submanifold $X := \{(x, y) \in \mathbb{R}^2 \mid px - qy \in \mathbb{Z}\} \subseteq \mathbb{R}^2$ from the definition of a torus link and notice that the group action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by component-wise addition restricts to a group action on X. We need to count the path-components of $X/\mathbb{Z}^2$.
In principle, the rest is combinatorics. To ease the notation, we note that

$$\begin{aligned} \pi_0(X/\mathbb{Z}^2) \times \pi_0(X/\mathbb{Z}^2) &\to \pi_0(X/\mathbb{Z}^2) \\ [x] + [y] &\mapsto [x + y] \end{aligned}$$

[11]The eagle-eyed among the readers will have noticed, that we now directly incorporated Convention 2.4 into the definition.

defines a group structure on the set of path-equivalence classes of $X/\mathbb{Z}^2$ and only consider the case $q \neq 0$ (the case $p \neq 0$ is analogous). Then

$$\begin{aligned} \mathbb{Z}/q\mathbb{Z} &\to \pi_0(X/\mathbb{Z}^2) \\ [n] &\mapsto \left[\left(\frac{n}{q}, 0\right)\right] \end{aligned}$$

is a well-defined group epimorphism with kernel $[p] \cdot (\mathbb{Z}/q\mathbb{Z})$. Hence,

$$\pi_0(X/\mathbb{Z}^2) \cong (\mathbb{Z}/q\mathbb{Z})/([p] \cdot (\mathbb{Z}/p\mathbb{Z})) \cong \mathbb{Z}/\gcd(p,q)\mathbb{Z}.$$

Shifting $X \subseteq \mathbb{R}^2$ an appropriate amount in x-direction shows that all components of $\mathrm{T}(p,q)$ are equivalent knots. This proves the second claim, since the component originating from the line in X going through $(0,0)$ is precisely the same as $\mathrm{T}(p/\gcd(p,q), q/\gcd(p,q))$. ■

One might ask when torus knots are trivial and more generally which torus knots are equivalent. We give a partial answer in the following proposition:

Proposition 3.2. *Let $p, q \in \mathbb{Z}$ not both zero.*

(1) *The links $\mathrm{T}(p,q)$, $\mathrm{T}(-p,-q)$, $\mathrm{T}(q,p)$ and $\mathrm{T}(-q,-p)$ are equivalent as oriented links.*

(2) *The link $\mathrm{T}(p,q)$ is trivial if $|q| \leq 1$ or $|p| \leq 1$.*

(3) *The mirror image of $\mathrm{T}(p,q)$ is $\mathrm{T}(p,-q)$.*

(4) *The link $\mathrm{T}(p,q)$ is invertible.*

Proof.

(1) By Lemma 2.2 the (p,q) cable of any unknot is equivalent to $\mathrm{T}(p,q)$. The claim therefore follows by considering the standard thickenings of unknots given by

$$\begin{aligned} \tau\colon S^1 \times \overline{B}^2 &\to S^3 \subseteq \mathbb{C}^2 \\ (x,y) &\mapsto \frac{(-x,-y)}{\|(-x,-y)\|} \end{aligned} \qquad \begin{aligned} \tau'\colon S^1 \times \overline{B}^2 &\to S^3 \subseteq \mathbb{C}^2 \\ (x,y) &\mapsto \frac{(y,x)}{\|(y,x)\|} \end{aligned}$$

and noticing that as oriented submanifolds

$$\tau(\mathrm{T}(p,q)) = \Theta(\mathrm{T}(-p,-q)) \qquad \text{and} \qquad \tau'(\mathrm{T}(p,q)) = \Theta(\mathrm{T}(q,p))$$

(2) By (1) we only need to show that $\mathrm{T}(0,q)$ and $\mathrm{T}(\pm 1, q)$ are trivial:
The link $\mathrm{T}(0,q)$ is the boundary of the smoothly embedded disks (with $k = 1, \ldots, q$)

$$\Theta\left(\{e^{2\pi \mathrm{i} \frac{k}{p}}\} \times \overline{B}^2\right) \subseteq S^3$$

The satellite Seifert surface for $\mathrm{T}(\pm 1, q)$ as a satellite of the unknot with thickening Θ is also a smoothly embedded disk.

(3) The orientation reversing diffeomorphism

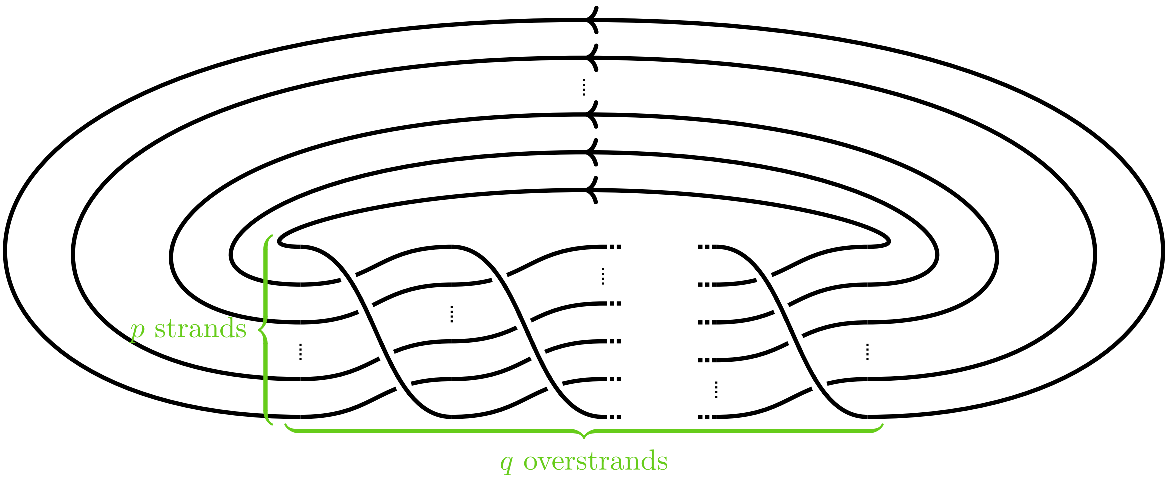
$$\begin{aligned} S^3 &\to S^3 \\ (x,y) &\mapsto (-x,y) \end{aligned}$$

maps $\mathrm{T}(p,q)$ to $\mathrm{T}(p,-q)$.

(4) By definition the oriented submanifolds $\mathrm{T}(p,q)$ and $\mathrm{T}(-p,-q)$ differ precisely in orientation, therefore, $\mathrm{T}(-p,-q)$ is the reversion of $\mathrm{T}(p,q)$. The claim now follows from (1). ■

To provide a more concrete way of viewing torus knots we construct a knot diagram for them:

Construction 3.3. Let $p, q \in \mathbb{Z}$ not both zero. In light of Proposition 3.2 we only consider the case $p > 0$. By projecting the torus vertically downwards onto an annulus, we then get a knot diagram for $\mathrm{T}(p, q)$. If $p > 0, q \geq 0$ this is given by

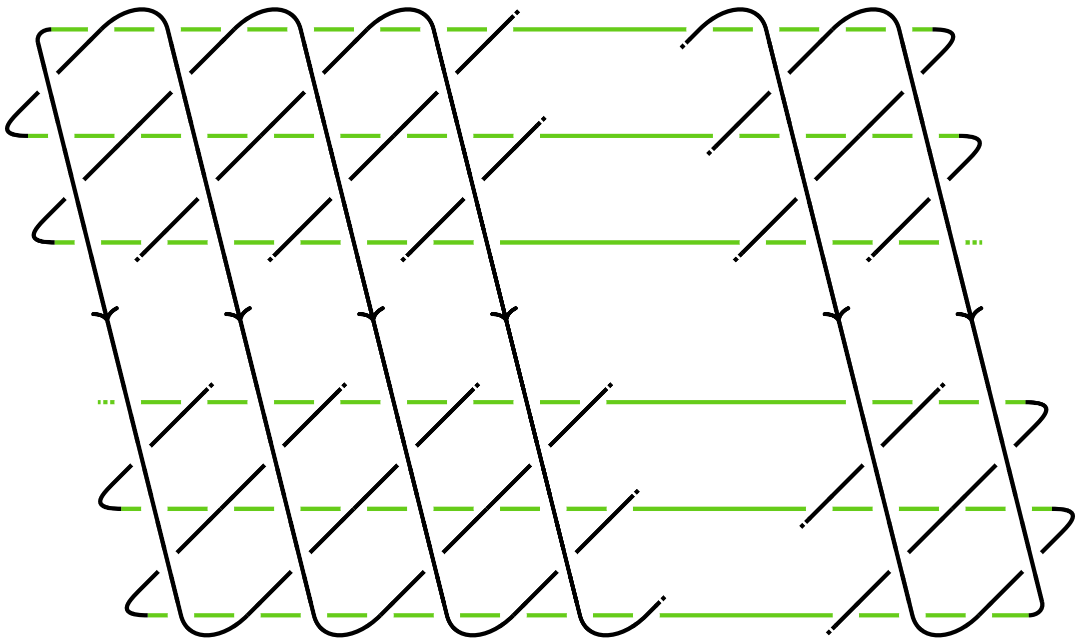


If $q < 0$ all crossings in the diagram have to be changed.

3.2 Alexander polynomial of torus knots

In this section, we want to find the Alexander polynomial of torus knots. Therefore we need to construct a Seifert surface and Seifert matrix for them:

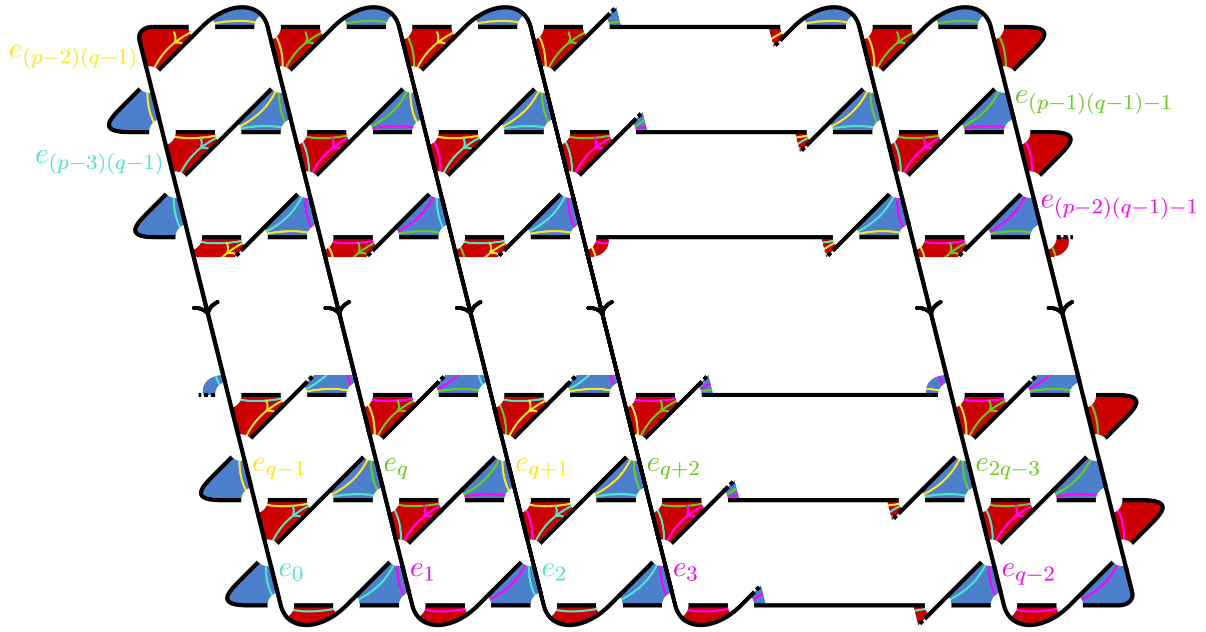
Construction 3.4. Let $p, q \in \mathbb{Z}$ not both zero. We only consider the case $p, q > 0$. Seifert surfaces for other values of p, q can be obtained using Proposition 3.2. In principle, we could directly apply Seifert's algorithm to the diagram constructed in Construction 3.3, but this would lead to a Seifert surface consisting of stacked discs which makes finding the harder. Instead, we shift the p parallel strands running at the top down, obtaining a new diagram for $\mathrm{T}(p, q)$.[12]



p horizontal lines and q downwards diagonals

[12]This technique for obtaining a "flat" Seifert surface from a braid word is further explored in [WC06, "flat style"]. For an abstract way to archive this without braids, see [OSS15, Proposition B.3.3].

This diagram leads to the following Seifert surface for $\mathrm{T}(p,q)$. We orient it such that the red side faces upwards:



The Seifert matrix from this Seifert surface relative to the basis $\{e_0, \ldots, e_{(p-1)(q-1)-1}\}$ is a $(q-1)\times(q-1)$ block matrix, where each block is $(p-1)\times(p-1)$:

$$\left(\begin{array}{c|c|c|c}
\begin{matrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{matrix} & & & \\ \hline
\begin{matrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{matrix} & \begin{matrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{matrix} & & \\ \hline
& \ddots & \ddots & \\ \hline
& & \begin{matrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{matrix} & \begin{matrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{matrix}
\end{array}\right)$$

Unfortunately, the Seifert matrix constructed in this way is somewhat unwieldy. We need to find a better way of describing it if we are to have any hope of calculating the Alexander polynomial from it:

Definition. Let $A = (a_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq m}}$ and $B = (b_{ij})_{\substack{1\leq i\leq n'\\ 1\leq j\leq m'}}$ be matrices over a ring. The $nn' \times mm'$ matrix

$$A \otimes B = \begin{pmatrix} a_{1,1}\cdot B & a_{1,2}\cdot B & \ldots & a_{1,m}\cdot B \\ a_{2,1}\cdot B & a_{2,2}\cdot B & \ldots & a_{2,m}\cdot B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}\cdot B & a_{1,2}\cdot B & \ldots & a_{n,m}\cdot B \end{pmatrix}$$

is the *Kronecker product* of A and B.

Example 3.5. Let $p, q \in \mathbb{N}$. The Seifert matrix for $\mathrm{T}(p,q)$ form Construction 3.4 is

$$\underbrace{\begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}}_{(q-1)\times(q-1)} \otimes \underbrace{\begin{pmatrix} -1 & & & & \\ 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}}_{(p-1)\times(p-1)}$$

We need a bit of theory about the Kronecker product. Far more can be said about it, as it acts as the tensor product of linear maps when considering matrices (see [JH91, Chapter 4.2]).

Lemma 3.6. *Let $A = (a_{ij})_{\substack{1\le i\le n\\1\le j\le m}}$, $B = (b_{ij})_{\substack{1\le i\le n'\\1\le j\le m'}}$, $C = (c_{ij})_{\substack{1\le i\le m\\1\le j\le l}}$ and $D = (d_{ij})_{\substack{1\le i\le m'\\1\le j\le l'}}$ be matrices over a ring R. Then*

(1) $\lambda \cdot (A \otimes B) = (\lambda A) \otimes B = A \otimes (\lambda B)$ *for all* $\lambda \in R$

(2) $(A \otimes B)^T = A^T \otimes B^T$

(3) $(A \otimes B)(C \otimes D) = AC \otimes BD$

(4) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ *if A and B are invertible*

Proof. Claims (1) and (2) follow by direct calculation, for (3) one additionally needs to consider the rules for multiplying block matrices. Claim (4) is a direct consequence of (3) and the fact that $1_n \otimes 1_m = 1_{nm}$. ∎

Furthermore, we remind the reader of roots of unity:

Definition. Let for all $n \in \mathbb{N}$ be $\zeta_n := e^{\frac{2\pi\,\mathrm{i}}{n}} \in \mathbb{C}$ be a primitive n-th root of unity. and

$$\mathrm{U}_n := \{\zeta \in \mathbb{C} \mid \zeta^n = 1\} = \langle \zeta_n \rangle \subseteq \mathbb{C}^*$$

be the group of n-th roots of unity.

Lemma 3.7. *Let $p, q \in \mathbb{N}$ with $d := \gcd(p,q)$. The group epimorphism*

$$\begin{aligned} \Phi\colon\ \mathrm{U}_p \times \mathrm{U}_q &\to \mathrm{U}_{\frac{pq}{d}} \\ (\zeta, \xi) &\mapsto \zeta \cdot \xi \end{aligned}$$

is d-to-1.

Proof. A direct calculation shows that Φ is a well-defined group homomorphism. Its kernel is given by

$$\ker(\Phi) = \left\{ \left(\zeta, \zeta^{-1}\right) \;\middle|\; \zeta \in U_d \right\}$$

since for $(\zeta, \xi) \in \mathrm{U}_p \times \mathrm{U}_q$

$$(\zeta, \xi) \in \ker(\Phi) \quad \Leftrightarrow \quad \zeta \cdot \xi = 1 \quad \Leftrightarrow \quad \zeta = \xi^{-1} \in \mathrm{U}_p \cap \mathrm{U}_q = \mathrm{U}_d$$

In particular $|\ker(\Phi)| = d$. By cardinality this implies that Φ is an epimorphism and

$$|\Phi^{-1}(\{\zeta\})| = d$$

for every $\zeta \in \mathrm{U}_{\frac{pq}{d}}$. ∎

Thus prepared, we can state and prove a formula for the Alexander polynomial of a torus knot:

Theorem 3.8. *Let $p, q \in \mathbb{Z}$ not both zero. If $p = 0$ or $q = 0$, $\Delta_{\mathrm{T}(p,q)}$ is trivial. If $p, q \neq 0$,*

$$\Delta_{\mathrm{T}(p,q)}(t) \doteq \prod_{\substack{i=1,\dots,|p|-1\\ j=1,\dots,|q|-1}} \left(t - \zeta_{|p|}^i \zeta_{|q|}^j\right) = \frac{\left(t^{\frac{|pq|}{d}} - 1\right)^d (t-1)}{(t^{|p|}-1)(t^{|q|}-1)}$$

where $d := \gcd(p, q) \in \mathbb{N}$.

Proof. By Proposition 3.2 $\mathrm{T}(p, q)$ is trivial if $p = 0$ or $q = 0$, which proves the first claim. Hence, we only need to consider the case $p, q \neq 0$. By Proposition 3.2

$$\mathrm{T}(p, q) = \mathrm{T}(-p, -q) = (\mathrm{T}(p, -q))^* = (\mathrm{T}(-p, q))^*$$

Therefore by Proposition 1.25

$$\Delta_{\mathrm{T}(p,q)}(t) = \Delta_{\mathrm{T}(-p,-q)}(t) = \Delta_{\mathrm{T}(p,-q)}(t) = \Delta_{\mathrm{T}(-p,q)}(t)$$

Hence we only need to consider the case $p, q > 0$.
We consider for all $n \in \mathbb{N}$ the $(n-1) \times (n-1)$ matrix

$$A_n := \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}$$

and start with a preliminary claim:

Claim. *For all $n \in \mathbb{N}$ there exists an invertible complex matrix X_n such that*

$$X_n \cdot \left(A_n^T\right)^{-1} A_n \cdot X_n^{-1} = \begin{pmatrix} -\zeta_n & & & & \\ & -\zeta_n^2 & & & \\ & & -\zeta_n^3 & & \\ & & & \ddots & \\ & & & & -\zeta_n^{n-1} \end{pmatrix}$$

Proof. We calculate that

$$\left(A_n^T\right)^{-1} A_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & \cdots & 1 \\ & & 1 & \cdots & 1 \\ & & & \ddots & 1 \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix} = \begin{pmatrix} & & & & 1 \\ -1 & & & & 1 \\ & -1 & & & 1 \\ & & \ddots & & \vdots \\ & & & -1 & 1 \end{pmatrix}$$

By Laplace extension in the last column its characteristic polynomial is therefore equal to

$$\chi_{A_n^{-1}A_n^T}(X) = \det \begin{pmatrix} X & & & & -1 \\ 1 & X & & & -1 \\ & \ddots & \ddots & & \vdots \\ & & 1 & X & -1 \\ & & & 1 & X-1 \end{pmatrix} = \sum_{i=0}^{n-1} (-1)^i \cdot X^{n-1-i} = \frac{X^n + (-1)^{n+1}}{X+1}$$

In particular, for all $z \in \mathbb{C} \setminus \{-1\}$

$$\chi_{A_n^{-1}A_n^T}(z) = 0 \quad \Leftrightarrow \quad z^n + (-1)^{n+1} = 0 \quad \Leftrightarrow \quad (-z)^n = 1 \quad \Leftrightarrow \quad -z \in \mathrm{U}_n \setminus \{1\}$$

which proves that the diagonalization of $\left(A_n^T\right)^{-1} A_n$ is as claimed. $\square$

By Construction 3.4 a Seifert matrix for $\mathrm{T}(p,q)$ is given by $M := A_q \otimes (-A_p)$. We can now calculate that

$$\det\left(M - t\cdot M^T\right) = \underbrace{\det\left(\left(\left(M^T\right)^{-1}\right)\right)}_{=(-1)^{(p-1)(q-1)}} \cdot \det\left(t\cdot M^T - M\right) = \det\left(t\cdot 1_{(p-1)(q-1)} - \left(M^T\right)^{-1} M\right)$$

$$\overset{\text{Lemma 3.6}}{=} \det\left(t\cdot 1_{(p-1)(q-1)} - \left(\left(A_q^T\right)^{-1} A_q \otimes \left(A_p^T\right)^{-1} A_p\right)\right)$$

$$= \det\left((X_q \otimes X_p)\cdot \left(t\cdot 1_{(p-1)(q-1)} - \left(\left(A_q^T\right)^{-1} A_q \otimes \left(A_p^T\right)^{-1} A_p\right)\right)\cdot (X_q\otimes X_p)^{-1}\right)$$

$$\overset{\text{Claim}}{=} \det\left(t\cdot 1_{(p-1)(q-1)} - \left(\begin{pmatrix} -\zeta_q & & & \\ & -\zeta_q^2 & & \\ & & \ddots & \\ & & & -\zeta_q^{q-1}\end{pmatrix} \otimes \begin{pmatrix} -\zeta_p & & & \\ & -\zeta_p^2 & & \\ & & \ddots & \\ & & & -\zeta_p^{p-1}\end{pmatrix}\right)\right)$$

$$= \prod_{\substack{i=1,\dots,p-1\\ j=1,\dots,q-1}} \left(t - \zeta_p^i\zeta_q^j\right)$$

which proves the first equality. For the second equality, we note that

$$\prod_{\substack{i=1,\dots,p-1\\ j=1,\dots,q-1}} \left(t - \zeta_p^i\zeta_q^j\right)\cdot (t^p-1)(t^q-1) = \prod_{\substack{i=1,\dots,p-1\\ j=1,\dots,q-1}} \left(t - \zeta_p^i\zeta_q^j\right) \prod_{i=1,\dots,p}\left(t-\zeta_p^i\right) \prod_{j=1,\dots,q}\left(t-\zeta_q^j\right)$$

$$= (t-1) \prod_{\substack{i=1,\dots,p-1\\ j=1,\dots,q}} \left(t - \zeta_p^i\zeta_q^j\right) \prod_{j=1,\dots,q}\left(t-\zeta_q^j\right) = (t-1)\prod_{\substack{i=1,\dots,p\\ j=1,\dots,q}} \left(t - \zeta_p^i\zeta_q^j\right)$$

$$\overset{\text{Lemma 3.7}}{=} (t-1)\cdot\left(t^{\frac{pq}{d}}-1\right)^d$$ ■

Remark. We have used in the last proof that the Alexander polynomial calculated from an invertible Seifert matrix M is given by the characteristic polynomial of $\left(M^T\right)^{-1}M$.
This can be further developed in the context of fibred knots – such as torus knots – since the Seifert matrix M originating from a fibre surface Σ is always invertible. The matrix $\left(M^T\right)^{-1}M$ then describes the monodromy of Σ, see [BZH13, Lemma 8.6] and [Sav12, Lemma 8.3].

As a corollary we obtain the genus of a torus knot:

Corollary 3.9. *Let $p, q \in \mathbb{Z}$ not both zero with $d := \gcd(p,q) \in \mathbb{N}$. The genus of $\mathrm{T}(p,q)$ is*

$$\begin{cases} \dfrac{(|p|-1)(|q|-1)-d+1}{2}, & \text{if } p,q \neq 0 \\ 0, & \text{if } p=0 \text{ or } q=0 \end{cases}$$

Proof. By Proposition 3.2 $\mathrm{T}(p,q)$ is a trivial knot or link if $p=0$ or $q=0$, which proves the second case. Hence, we only need to consider the case $p,q\neq 0$. By Theorem 3.8

$$\mathrm{br}\left(\Delta_{\mathrm{T}(p,q)}(t)\right) = (|p|-1)(|q|-1)$$

By Lemma 3.1 $\mathrm{T}(p,q)$ is a d-component link. By Proposition 1.27 the genus of $\mathrm{T}(p,q)$ is therefore greater than or equal to

$$\frac{\mathrm{br}\left(\Delta_{\mathrm{T}(p,q)}(t)\right)-d+1}{2} = \frac{(|p|-1)(|q|-1)-d+1}{2}$$

This proves the claim, since the Seifert surface constructed in Construction 3.4 has $2|pq|-2$ discs and $3|pq|-|p|-|q|-2$ bands with d boundary components and thereby genus

$$\frac{(3|pq|-|p|-|q|-2)-(2|pq|-2)+1-(d-1)}{2} = \frac{(|p|-1)(|q|-1)-d+1}{2}$$ ■

3.3 Classification of torus knots

The aim of this last section is to prove that between torus knots only the equivalences described in Proposition 3.2 hold. Part of this is to show that torus knots are chiral:

Proposition 3.10. *Any non-trivial torus link is chiral.*

Proof. see [Mur08, Theorem 7.4.2].
The idea of the proof is to show that the signature of a non-trivial torus link is non-zero. This proves the claim, as it follows from Proposition 1.17 that the signature of an amphichiral link must be 0. ■

With this proposition in place, we can give a complete classification of torus knots:

Theorem 3.11. *Let $p, q, p', q' \in \mathbb{Z}$ not both zero.*

(1) *Two non-trivial torus links $\mathrm{T}(p, q)$ and $\mathrm{T}(p', q')$ are equivalent oriented links if and only if*

$$(p', q') \in \{(p, q), (q, p), (-p, -q), (-q, -p)\}$$

(2) *The link $\mathrm{T}(p, q)$ is trivial if and only if $|p| \leq 1$ or $|q| \leq 1$.*

Proof. By Proposition 3.2 we only need to prove the "only if" direction of both claims:

(2) Assume $\mathrm{T}(p, q)$ is trivial. The claim is true of $p = 0$ or $q = 0$, hence we only need to consider $p, q \neq 0$. By Theorem 3.8

$$\Delta_{\mathrm{T}(p,q)}(t) \doteq \prod_{\substack{i=1,\dots,|p|-1 \\ j=1,\dots,|q|-1}} \left(t - \zeta_{|p|}^i \zeta_{|q|}^j\right)$$

This product is non-zero. Since $\Delta_{\mathrm{T}(p,q)}(t)$ is trivial we must therefore have $\Delta_{\mathrm{T}(p,q)}(t) \doteq 1$. This implies that the product is empty, i.e. $|p| = 1$ or $|q| = 1$.

(1) As we only consider non-trivial links $|p|, |q| \geq 2$ by (2).
Assume $\mathrm{T}(p, q)$ and $\mathrm{T}(p', q')$ are equivalent as oriented links. Let $d := \gcd(p, q)$ and $d' := \gcd(p', q')$. By Lemma 3.1 $d = d'$ is the number of components of $\mathrm{T}(p, q)$ (or $\mathrm{T}(p', q')$). By Theorem 3.8

$$|pq| - |p| - |q| - 1 = \mathrm{br}\Big(\mathrm{T}_{(p,q)}(t)\Big) = \mathrm{br}\Big(\mathrm{T}_{(p',q')}(t)\Big) = |p'q'| - |p'| - |q'| - 1$$
$$\frac{|pq|}{d} = \max_{n \in \mathbb{N}} \left\{\Delta_{\mathrm{T}(p,q)}(\zeta_n) = 0\right\} = \max_{n \in \mathbb{N}} \left\{\Delta_{\mathrm{T}(p',q')}(\zeta_n) = 0\right\} = \frac{|p'q'|}{d'}$$

Therefore, $|p| \cdot |q| = |p'| \cdot |q'|$ and $|p| + |q| = |p'| + |q'|$. This implies that

$$\begin{aligned}(X - |p|)(X - |q|) &= X^2 - (|\,p\,| + |\,q\,|)X + |\,p\;q\,| \\ &= X^2 - (|p'| + |q'|)X + |p'q'| = (X - |p'|)(X - |q'|)\end{aligned}$$

i.e. $\{|p|, |q|\} = \{|p'|, |q'|\}$. By Proposition 3.10 $\mathrm{T}(p, q)$ is not equivalent to $\mathrm{T}(p, -q)$. The claim now follows from Proposition 3.2. ■

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