I  Theorem

Let $A$ be a factorial ring. Then the polynomial ring $A[X]$ in one variable is factorial. Its prime elements are the primes of $A$ and polynomials in $A[X]$ which are irreducible in $K[X]$ and have content $1$.

**Remark:** Let $P = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}[X]$ be a polynomial with integer coefficients. The content of $P$ is the greatest common divisor of the coefficients of $P$. It’s denoted by $c(P)$.

II  Criteria for irreducibility

2.1

Let $A$ be a factorial ring. Let $K$ be its quotient field. Let $f(X) = a_n X^n + \ldots + a_1 X + a_0$ be a polynomial of degree $n \geq 1$ in $A[X]$. Let $p$ be a prime of $A$, and assume:
- $a_n \not\equiv 0 \pmod{p}$
- $a_i \equiv 0 \pmod{p}$ $\forall i < n$
- $a_0 \not\equiv 0 \pmod{p^2}$

Then $f(X)$ is irreducible in $K[X]$.

2.1.1 Example

Let $a$ be a non-zero square-free integer $\neq \pm 1$. Then for any integer $\geq 1$, the polynomial $X^n - a$ is irreducible over $\mathbb{Q}$.

The polynomials $3X^5 - 15$ and $2X^{10} - 21$ are irreducible over $\mathbb{Q}$.

2.1.2 another one

Let $p$ be a prime number. Then the polynomial $f(X) = X^{p-1} + \ldots + 1$ is irreducible over $\mathbb{Q}$.

2.1.3 remark

Let $E$ be a field and $t$ an element of some field containing $E$ such that $t$ is transcendental over $E$ ($\ldots$)

($\ldots$) This comes from the fact that the ring $A = E[t]$ is factorial and that $t$ is a prime in it.

Lineare Algebra II - remarks 1
2.2

Let \( p \) be a prime number. \( X^p - X - 1 \) is irreducible over the field \( \mathbb{Z}/p\mathbb{Z} \). Hence \( X^p - X - 1 \) is irreducible over \( \mathbb{Q} \supseteq \mathbb{Z}/p\mathbb{Z} \).

Similarly:

\( X^5 - 5X^4 - 6X - 1 \) is irreducible over \( \mathbb{Q} \).

2.3

Let \( A \) be a factorial ring and \( K \) its quotient field. Let \( f(X) = a_nX^n + \ldots + a_0 \in A[X] \).

Let \( \alpha \in K \) be a root of \( f \) with \( \alpha = \frac{b}{d} \) expressed with \( b, d \in A \) and \( b, d \) relatively prime.

Then \( b \mid a_0 \) and \( d \mid a_n \).

In particular, if the leading coefficient \( a_n \) is 1, then a root \( \alpha \) must lie in \( A \) and divides \( a_0 \).

### III tensor products of algebras

In this section, we again let \( R \) be a commutative ring. By an \( R \)-algebra we mean a ring homomorphism \( R \rightarrow A \) into a Ring \( A \) such that the image of \( R \) is contained in the center of \( A \).

Let \( A, B \) be \( R \)-algebras. We shall make \( A \otimes B \) into an \( R \)-algebra. Given \( (a, b) \in A \times B \) we have an \( R \)-bilinear map

\[
M_{a,b} : A \times B \rightarrow A \otimes B \text{ such that } M_{a,b}(a', b') = aa' \otimes bb'
\]

Hence \( M_{a,b} \) induces an \( R \)-linear map \( m_{a,b} : A \otimes B \rightarrow A \otimes B \) such that \( m_{a,b}(a', b') = aa' \otimes bb' \).

But \( m_{a,b} \) depends bilinearly on \( a \) and \( b \), so we obtain finally a unique \( R \)-bilinear map

\[
A \otimes B \times A \otimes B \rightarrow A \otimes B
\]

such that \( (a \otimes b)(a' \otimes b') = aa' \otimes bb' \). This map is obviously associative, and we have a natural ring homomorphism

\( R \rightarrow A \otimes B \) given by \( c \mapsto 1 \otimes c = c \otimes 1 \).

Thus \( A \otimes B \) is an \( R \)-algebra, called the ordinary tensor product.