Math 1210-009 Fall 2013

Third Midterm Examination

18th November 2013

Name:

- No cell phones, computers, etc.
- No cheating.
- No notes, cheat sheets, books, etc.
- Write your name on each page.
- Show your work to get full credit.
- Make sure that what you write down is mathematically correct, e.g. don't forget equal signs etc.

	1	2	3: extra credit	$\sum$	%
Possible points	25	25	10	50 + 10	100 + 20
Your points					

1 Using the techniques learned in class, graph the function given by the equation:

$$f(x) = -\frac{1}{4}x^4 + 2x^2.$$

Domain: defined everywhere $\Rightarrow D_f = \mathbb{R}$

Symmetries: $f(-x) = -\frac{1}{4}(-x)^4 + 2(-x)^2 = -\frac{1}{4}x^4 + 2x^2 = f(x)$

$\Rightarrow$ symmetric with respect to y-axis.

y-intercept: $f(0) = 0 \quad \Rightarrow \quad P_1 = (0,0)$

Zeros: $0 = f(x) = -\frac{1}{4}x^4 + 2x^2 = x^2\left(\frac{1}{2}x + \sqrt{2}\right)\left(-\frac{1}{2}x + \sqrt{2}\right)$

$\Rightarrow x_1 = 0, \; x_2 = -2\sqrt{2} \quad x_3 = 2\sqrt{2} \quad \Rightarrow \quad P_2 = (-2\sqrt{2}, 0), \; P_3 = (2\sqrt{2}, 0)$

Derivatives: $f'(x) = -x^3 + 4x = x(-x+2)(x+2)$

$f''(x) = -3x^2 + 4 = (-\sqrt{3}x + 2)(-\sqrt{3}x + 2)$

$f'''(x) = -6x$

Critical points: No endpoints.

No singular points.

Stationary points: $0 = f'(x) = x(-x+2)(x+2) \Rightarrow x_1 = 0$

$\Rightarrow x_1 = 0, \; x_4 = 2, \; x_5 = -2$

$y_4 = f(x_4) = f(2) = -\frac{1}{4}\cdot 16 + 2\cdot 4 = 4 \qquad P_4 = (2,4)$

$y_5 = f(x_5) = f(-2) = 4 \qquad P_5 = (-2,4)$

Maxima or minima: $f''(x_1) = f''(0) = 4 > 0 \quad \Rightarrow$ MIN : P_1

$f''(x_4) = f''(2) = -8 < 0 \quad \Rightarrow$ MAX : P_4

$f''(x_5) = f''(-2) = -8 < 0 \quad \Rightarrow$ MAX : P_5

Candidates for inflection points: f'' exists everywhere

$0 = f''(x) = (-\sqrt{3}x + 2)(\sqrt{3}x + 2)$

$\Rightarrow x_6 = \frac{2}{\sqrt{3}}, \; x_7 = -\frac{2}{\sqrt{3}}$

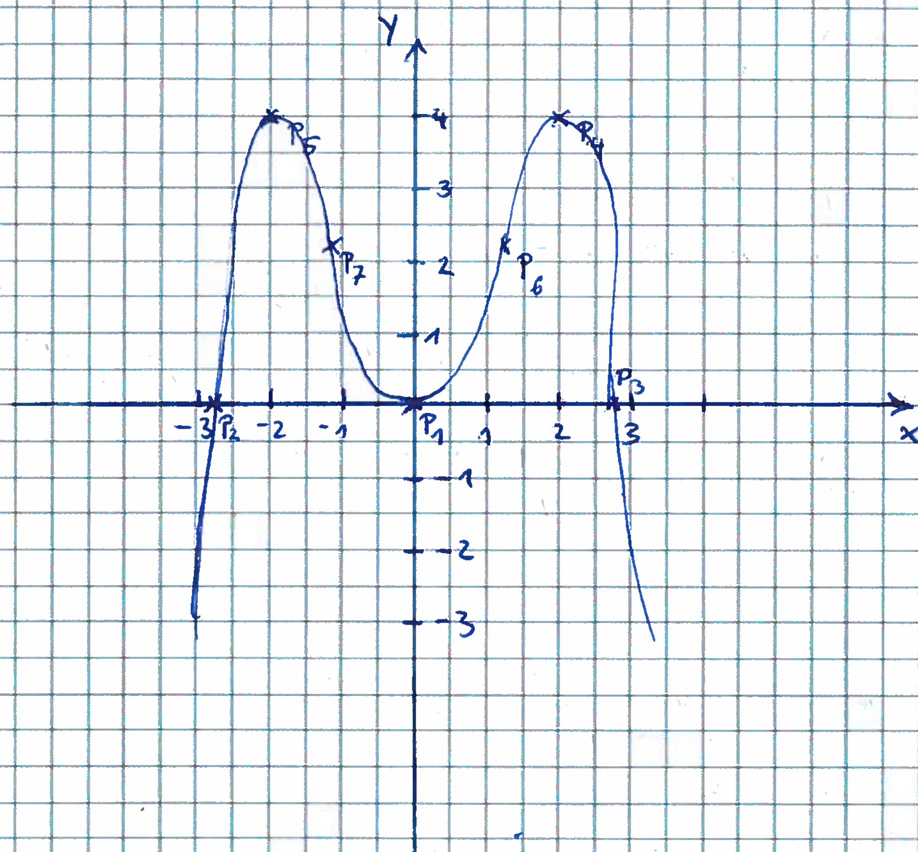
$y_6 = f(x_6) = -\frac{1}{4}\cdot\frac{16}{9} + 2\cdot\frac{4}{3} = \frac{20}{9} \qquad P_6 = \left(\frac{2}{\sqrt{3}}, \frac{20}{9}\right)$

$y_7 = f(x_7) = \frac{20}{9} \qquad P_7 = \left(-\frac{2}{\sqrt{3}}, \frac{20}{9}\right)$

Are they inflection points: $f'''(x_6) = -6 \cdot \frac{2}{\sqrt{3}} = -4\sqrt{3} \neq 0$

$f'''(x_7) = -6\left(-\frac{2}{\sqrt{3}}\right) = 4\sqrt{3} \neq 0$

No assymptotes.



2 Using the techniques learned in class, graph the function given by the equation:

$$f(x) = \frac{x^2}{x^2 - 16}.$$

Domain: $f(x) = \frac{x^2}{(x+4)(x-4)}$ => not defined at $4, -4$ => $D_f : \mathbb{R} \setminus \{-4, 4\}$

Symmetries: $f(-x) = \frac{(-x)^2}{(-x)^2 - 16} = \frac{x^2}{x^2 - 16}$ => symmetric with respect to y-axis

y-intercept: $f(0) = 0$ => $P_1 = (0, 0)$

Zeros: $0 = f(x) = \frac{x^2}{x^2 - 16}$ <=> $0 = x^2$ => only zero $= P_1$

Derivatives: $f'(x) = \frac{(x^2-16) \cdot 2x - x^2 \cdot 2x}{(x^2-16)^2} = -\frac{32x}{(x^2-16)^2}$

$f''(x) = -\frac{(x^2-16)^2 \cdot 32 - 32x \cdot 2(x^2-16) \cdot 2x}{(x^2-16)^4} = \frac{512 + 96x^2}{(x^2-16)^3}$

$f'''(x) = \frac{(x^2-16)^3 \cdot 192 - (512 + 96x^2) \cdot 3(x^2-16)^2 \cdot 2x}{(x^2-16)^6} = \frac{(-3x^3 + x^2 - 2^4 x - 2^4) \cdot 3 \cdot 2^6}{(x^2-16)^4}$

Critical points: No endpoints.
No singular points.
Stationary points: $0 = f'(x) = -\frac{32x}{(x^2-16)^2}$ <=> $x = 0$ => only stationary point $= P_1$

Maximum or minimum: $f''(x_1) = f''(0) = \frac{512}{(-16)^3} = -\frac{1}{8} < 0$ => MAX

Candidates for inflection points: f'' exists everywhere on D_f

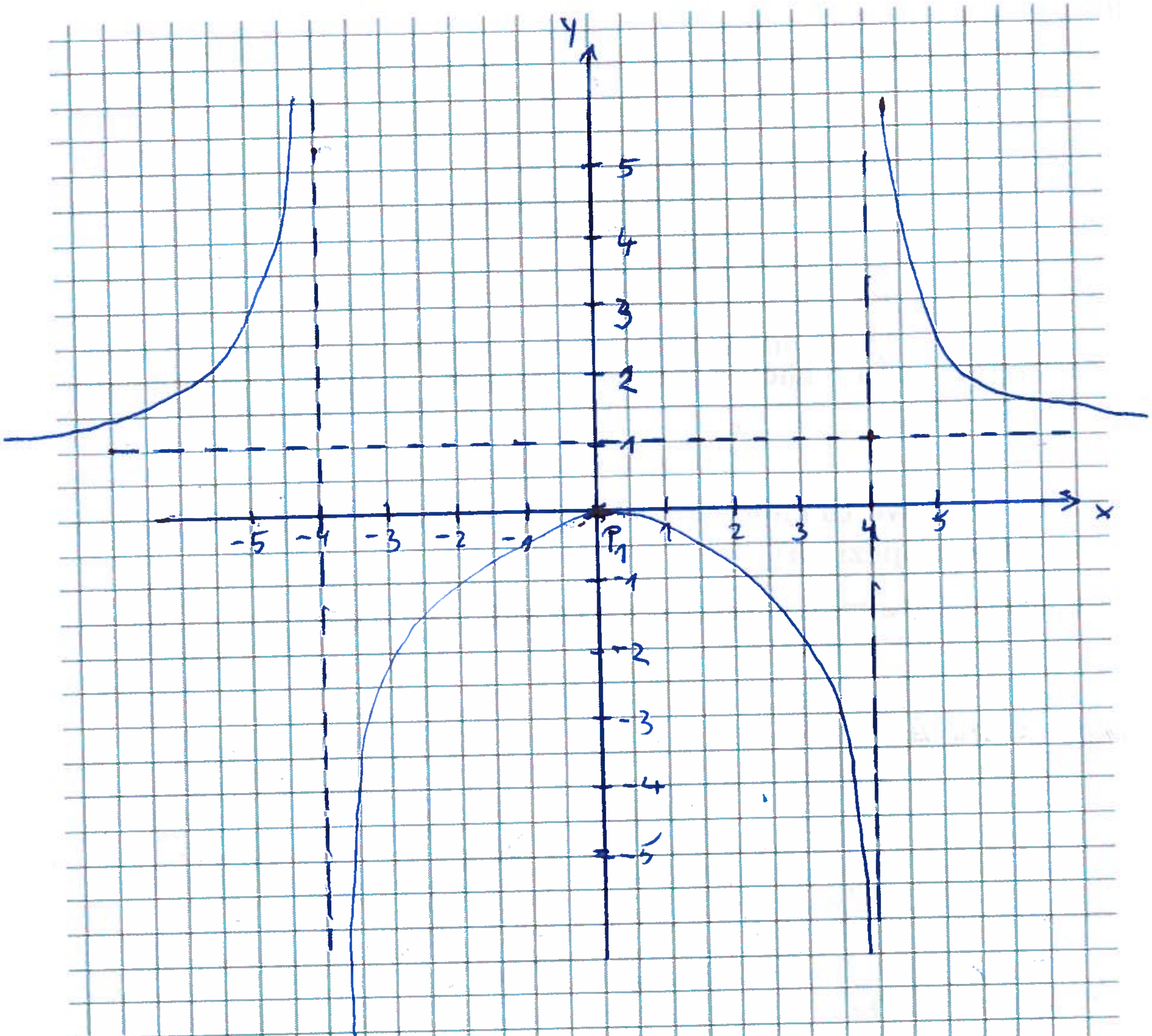
$0 = f''(x) = \frac{512 + 96x^2}{(x^2-16)^3}$ <=> $512 + 96x^2 = 0$ not possible => no inflection points

Asymptotes: vertical asymptotes: $\lim_{x \to -4^-} \frac{x^2}{x^2-16} = +\infty$ $\quad \lim_{x \to -4^+} \frac{x^2}{x^2-16} = -\infty$

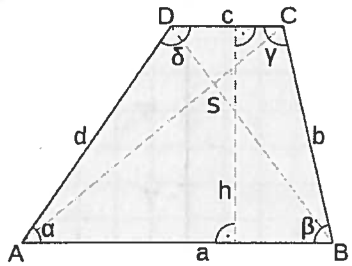
$\lim_{x \to 4^-} \frac{x^2}{x^2-16} = -\infty$ $\quad \lim_{x \to 4^+} \frac{x^2}{x^2-16} = +\infty$

horizontal asymptote: $\lim_{x \to +\infty} \frac{x^2}{x^2-16} = \lim_{x \to +\infty} \frac{1}{1 - \frac{16}{x^2}} = 1$

$\lim_{x \to -\infty} \frac{x^2}{x^2-16} = 1$


y
x
5
4
3
2
1
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
1
2
3
4
5
P_1

3 Consider a trapezium:



The angles α and β should be $45°$. The sides b, c and d should sum up to 28 meters:

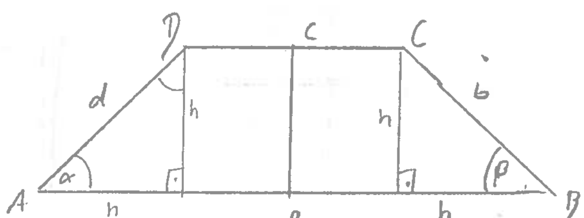
$$b + c + d = 28m$$

How do we have to choose the height h in order to maximise the area of the trapezium?

Hint: the area of a trapezium is calculated with the formula

$$A = \frac{a+c}{2} \cdot h.$$

Since α and β are $45°$



1. $b = d = \sqrt{h^2 + h^2} = \sqrt{2}\, h$

2. $28m = b + c + d = 2\sqrt{2}\, h + c \quad \Rightarrow \quad c = 28m - 2\sqrt{2}\, h$

3. $a = 2h + c =$

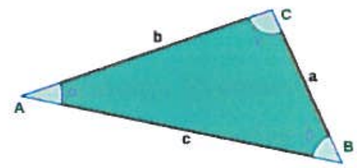
$\Rightarrow \frac{a+c}{2} = \frac{2h + c + c}{2} = h + c = h + 28m - 2\sqrt{2}\, h = 28m + (1 - 2\sqrt{2})h$

$\Rightarrow A(h) = \frac{a+c}{2} h = \left(28m + (1 - 2\sqrt{2})h\right)h = 28m \cdot h + (1 - 2\sqrt{2})h^2$

$A'(h) = 28m + 2(1 - 2\sqrt{2})\, h = 0 \quad \Rightarrow \quad h = \frac{-28m}{2(1 - 2\sqrt{2})} = 7{,}657\ m$

$A''(h) = 2(1 - 2\sqrt{2}) = -3{,}657 < 0 \quad \Rightarrow$ MAX. at $h = 7.657\, m$

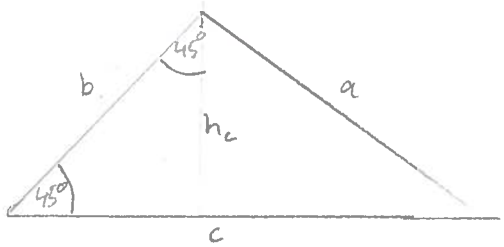
4 Consider all triangles with one fixed angle $\alpha = 45°$ and fixed sum $b + c = 5$.



How do we have to choose c to maximize the area?

Hint: the area of a triangle is

$$A = \frac{1}{2} c \cdot h_c.$$



1. $b^2 = 2h^2 \Rightarrow h = \frac{1}{\sqrt{2}} b$
2. $b = 5 - c \Rightarrow h = \frac{1}{\sqrt{2}}(5-c)$

$A(c) = \frac{1}{2} c \cdot h = \frac{1}{2} c \cdot \frac{1}{\sqrt{2}}(5-c) = \frac{5}{2\sqrt{2}} c - \frac{1}{2\sqrt{2}} c^2 = \frac{1}{2\sqrt{2}} c(5-c) = \frac{1}{2\sqrt{2}}(5c - c^2)$

$A'(c) = \frac{1}{2\sqrt{2}}(5 - 2c) = 0 \Rightarrow c = \frac{5}{2} = 2{,}5$

$A''(c) = \frac{1}{2\sqrt{2}}(-2) = -\frac{1}{\sqrt{2}} < 0 \Rightarrow$ MAX at $c = 2.5$