# Math 1210-009 Fall 2013

First Midterm Examination

23rd September 2013

**Name:**

- No cell phones, computers, etc.
- No cheating.
- No notes, cheat sheets, books, etc.
- Write your name on each page.
- Show your work to get full credit.

| | 1 | 2 | 3 | 4 | $\sum$ | % |
|---|---|---|---|---|---|---|
| Possible points | 20 | 10 | 10 | 10 | 50 | 100 |
| Your points | | | | | | |

1. In each of the following problems find the indicated limit or state that it does not exist.

(a)

$$\lim_{x\to+\infty} \frac{2x^2-2x+3}{2-x^2}$$

$$\lim_{x\to\infty} \frac{2x^2-2x+3}{2-x^2} = \lim_{x\to\infty} \frac{2-\frac{2}{x}+\frac{3}{x^2}}{\frac{2}{x^2}-1} = \frac{2}{-1} = -2$$

or

$$\lim_{x\to\infty} \frac{2x^2-2x+3}{2-x^2} = \lim_{x\to 0} \frac{2\frac{1}{x^2}-2\frac{1}{x}+3}{2-\frac{1}{x^2}} =$$

$$= \lim_{x\to 0} \frac{2-2x+3x^2}{2x^2-1} = \underline{\underline{-2}}$$

(b)

$$\lim_{x\to+\infty} \sqrt{\frac{x}{4x-9}}$$

$$\lim_{x\to\infty} \sqrt{\frac{x}{4x-9}} = \sqrt{\lim_{x\to\infty} \frac{x}{4x-9}} =$$

$$= \sqrt{\lim_{x\to\infty} \frac{1}{4-\frac{9}{x}}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

or

$$\lim_{x\to\infty} \sqrt{\frac{x}{4x-9}} = \sqrt{\lim_{x\to\infty} \frac{x}{4x-9}} =$$

$$= \sqrt{\lim_{x\to 0} \frac{\frac{1}{x}}{\frac{4}{x}-9}} =$$

$$= \sqrt{\lim_{x\to 0} \frac{1}{4-9x}} = \sqrt{\frac{1}{4}} = \underline{\frac{1}{2}}$$

(c)

$$\lim_{x\to-\infty}\frac{\sin x}{x}$$

We know: $\forall x \in \mathbb{R}: \sin x \in [-1, 1]$, that is $\sin x$ is bounded.

$$\lim_{x\to-\infty}\frac{\sin x}{x} = \lim_{x\to-\infty}\left(\frac{1}{x}\cdot\sin x\right) = \underline{\underline{0}}$$

because $\lim_{x\to-\infty}\frac{1}{x} = 0$ and $\sin x$ is bounded.

(d)

$$\lim_{x\to 0}\frac{\sin(5x)}{\sin(3x)}$$

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x} = \lim_{x\to 0}\frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}} =$$

$$= \lim_{x\to 0}\frac{5\frac{\sin 5x}{5x}}{3\frac{\sin 3x}{3x}} =$$

$$= \frac{5\cdot\lim_{x\to 0}\frac{\sin 5x}{5x}}{3\cdot\lim_{x\to 0}\frac{\sin 3x}{3x}} = \underline{\underline{\frac{5}{3}}}$$

2. In each of the following problems check if the limit exists by calculating the right and left handed limits.

   (a)
$$\lim_{x\to 1} \frac{|x^4-1|}{x^2-1}$$

$$\lim_{x\to 1^+} \frac{|x^4-1|}{x^2-1} = \lim_{x\to 1^+} \frac{x^4-1}{x^2-1} = \lim_{x\to 1^+} \frac{(x^2-1)(x^2+1)}{(x^2-1)} = \cancel{2}$$

$$= \lim_{x\to 1^+} x^2+1 = \underline{\underline{2}}$$

$$\lim_{x\to 1^-} \frac{|x^4-1|}{x^2-1} = \lim_{x\to 1^-} \frac{-(x^4-1)}{x^2-1} = \lim_{x\to 1^-} \frac{-(x^2-1)(x^2+1)}{x^2-1} =$$

$$= \lim_{x\to 1^-} -(x^2-1) = \underline{\underline{-2}}$$

$\Rightarrow$ RHL $\neq$ LHL $\Rightarrow$ the limit does not exist

   (b)
$$\lim_{x\to 1} \frac{x^4-1}{x^2-1}$$

$$\lim_{x\to 1^+} \frac{x^4-1}{x^2-1} = \lim_{x\to 1^+} \frac{(x^2-1)(x^2+1)}{x^2-1} = \lim_{x\to 1^+} x^2+1 = \underline{\underline{2}}$$

$$\lim_{x\to 1^-} \frac{x^4-1}{x^2-1} = \lim_{x\to 1^-} \frac{(x^2-1)(x^2+1)}{x^2-1} = \lim_{x\to 1^-} x^2+1 = \underline{\underline{2}}$$

$\Rightarrow$ RHL $=$ LHL $\Rightarrow$ the limit exists

3. Determine the points, where the following functions are not contiuous. Determine if the discontinuities are removable and if so, give a continuation of the function at these points.

(a)

$$f(x) = \frac{2x^2 - 98}{x - 7}$$

$f(x)$ is not continuous ~~at~~ at $x = 7$.

Aside: $\frac{2x^2-98}{x-7} = \frac{2(x^2-49)}{x-7} = \frac{2(x+7)(x-7)}{x-7} = 2(x+7)$

The function can be simplified. ~~to~~

Define $\tilde{f}(x) = 2(x+7)$

We have that $\tilde{f}(x) = f(x) \quad \forall x \neq 7$

and $\tilde{f}(x)$ is defined everywhere, thus it is a continuation of $f$.

(b)

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

$f(x)$ is not continuous at $x = 0$.

We have $\lim_{x \to 0^+} x \sin\frac{1}{x} = 0$ as $\lim_{x \to 0^+} x = 0$ and $\sin\frac{1}{x} \in [-1, 1]$.

$\lim_{x \to 0^-} x \sin\frac{1}{x} = 0$ for the same reason.

$\Rightarrow$ The limit ~~$\lim_{x \to 0}$~~ exists and is $\lim_{x \to 0} x \sin\frac{1}{x} = 0$

~~By~~ Thus $f(x)$ can be continued at $0$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

4. Use the Intermediate Value Theorem to answer the following questions.

(a) Show that the equation $x^3 - 2x^2 + \frac{1}{2} = 0$ has a solution between 1 and 2.

Let $f(x) = x^3 - 2x^2 + 1$, This is continuous on the interval $[1, 2]$.

~~Then~~ Then $f(1) = 1 - 2 + \frac{1}{2} = -\frac{1}{2} < 0$

$f(2) = 8 - 8 + \frac{1}{2} = \frac{1}{2} > 0$.

By the Intermediate Value Theorem ther is a number $c \in (1,2)$ such that $f(c) = 0 \in (f(1), f(2))$

Thus $c$ would be a solution for the equation.

(b) Why has the equation $x^2 + 1 = 0$ **no** solution? in real numbers?

As $x^2 \geq 0 \quad \forall x \in \mathbb{R}$, we know that

$x^2 + 1 > 0 \quad \forall x \in \mathbb{R}$. ~~and never~~

Thus $\nexists\, x \in \mathbb{R}$ such that $x^2 + 1 = 0$.