Graphing functions

Possible problem:

Graph the function $y = f(x)$ using the methods discussed in class.

For differentiable functions:

- Generally: f' determines the change of f, f'' determines the change of f', f''' determines the change of f'', etc.
- The first derivative gives the slope at a point. If the first derivative is zero, the slope is zero. This is a candidate for an extrem point.
- The slope is closely related to the monotonicity of the function. If $f'(x) > 0$ on an **interval** the function is increasing. If $f'(x) < 0$ on an **interval** the function is decreasing.
- The second derivative determines the concavity or curvature (or the change of the slope) of a function. If $f''(x) > 0$ on an interval, then f is convex. If $f''(x) < 0$ on an interval, then f is concave. If $f''(x) = 0$ then there is no curvature. The points where $f''(x) = 0$ are candidates for inflection points.
- The third derivative determines the change of concavity. Only if $f'''(x) \neq 0$ can we have a change from concave to convex and vice versa.

Important points of a function are:

Zeros Points, when the graph of f crosse the x-axis: $f(x) = 0$.

y-intercept Point when the graph crosses the y.axis: $y = f(0)$.

Extreme points Points when the graph attains a local maximum. Candidates are the critical points: stationary points ($f'(x) = 0$), singular points ($f'(x)$ does not exist at this point), endpoints.

Maxima If possible check with second derivative: $f''(x) < 0$. If this is not possible, check with first derivative or check the function itself in an interval around the critical point.

Minima If possible check with second derivative: $f'(x) > 0$. If this is not possible, check with first derivative or check the function itself in an interval around the critical point.

Inflection points Points, when the graph changes concavity/curvature. Candidates are: points x such that $f''(x) = 0$, points such that $f''(x)$ does not exist. In the first case, we have to have that $f'''(x) \neq 0$

Saddle points These are inlfection points such that in addition the slope is 0. That is in addition $f'(x) = 0$.

In order to graph a function one tries to calculate "enough" points and then plot the function at these points. Among those points should be the important points listed above.

Here is a general method:

1. Check the domain and range of the function. Draw the coordinate system accordingly.
2. Test for symmetry with respect to y axis and origin. Is the function even or odd or nothing?
3. Find the y-intercept: $y = f(0)$. This gives a point: $P = (0, f(0))$
4. Find the zeros: $f(x) = 0$. This gives points: $P_{i_0} = (x_{i_0}, 0)$
5. Calculate the first three derivatives if possible.

6. Find the critical points x_{i_1}: end points, stationary points ($f'(x) = 0$), singular points ($f'(x)$ does not exist). Plug them back into the original function $y_{i_1} = f(x_{i_1})$. This gives points $P_{i_1} = (x_{i_1}, y_{i_1})$.
7. Check wether these points are maxima or minima or neither.
   (a) For end points: Use the first derivative test if the first derivative exists in an interval around this end point. If it does not exist check the function itself around the end point.
   (b) For stationary points: Use the second derivative test, if the second derivative exists at this point. If not, use the first derivative test in an interval around this point.
   (c) For singular points: Use the first derivative test if the first derivative exists in an interval around this point. If it does not exist check the function itself around the point.
8. Find the candidates for inflection points x_{i_2}: points, when $f''(x_{i_2}) = 0$ or does not exist. Plug them back into the original function $y_{i_2} = f(x_{i_2})$. This gives points $P_{i_2} = (x_{i_2}, y_{i_2})$.
9. Check if they are really inflection points.
   (a) When $f''(x_{i_2}) = 0$ and if f''' exists at this point and $f'''(x_{i_2}) \neq 0$
   (b) When $f''(x_{i_2}) = 0$ but f''' does not exist at this point, one has to check f'' in an interval around this point. Does it change signs?
   (c) When $f''(x_{i_2})$ does not exist, but exists in an interval around x_{i_2}, one has to check f'' in an interval around this point. Does it change signs?
10. Check if infelction points are saddle points: is in addition $f'(x_{i_2}) = 0$?
11. For rational functions (among others) find asymptotes.

    Poles or vertical asymptotes: points x_{i_3} where $f(x)$ is not defined and $\lim_{x\to x_{i_3}} f(x) = \pm\infty$ for left and/or right handed limits.

    Horizontal asymototes exist when $\lim_{x\to\infty} f(x) = c$ or $\lim_{x\to-\infty} f(x) = c$ where c is a real number.

    Oblique asymptotes for rational functions. If the degree of the numerator is bigger than the degree of the denominator, do polynomial division with remainder, i.e. divide the numerator by the denominator with longdivision, until there is a remainder left, whose degree is smaller than the degree of the denominator. The remainder will go to 0 as $x \to \pm\infty$. The first part gives an asymptotic function and in the case when the degrees of denominator and numerator differ by 1 only, it is a line and gives an oblique asymptote.
12. Plot the calculated points, and if necessary additional points.
13. Draw the asymptotes.
14. Sketch the graph.

Use a ruler and make it as accurate as possible in order to avoid misunderstandings.

Practical problems

These are optimisation problems. Such as: under certain circumstances (for example the perimeter is fixed, the shape is fixed, and angel is fixed, an area is fixed, a volume is fixed, etc.) find the best way to do something, maximise the area, maximise the volume, etc. Because of the very general nature of these problems, there cannot be a recipe that works for all. Instead one has to use common sense and be creative.

A possible step-by-step method might be the following:

1. Read the instructions carefully.

2. Draw a picture to illustrate the problem.
3. Assign appropriate variables to the important quantities.
4. Write a formula for the quantity or objective function to be maximised or minimised in terms of the variables of the previous step.
5. Use the conditions to eliminate all but one of the variables.
6. Find the critical points.
7. Determine if these are maxima or minima.
8. Check if they make sense (e.g. there are no negative lengths).