Functions

Definition 1. A rule that assignes to each element in the domain exactly one element in the target is called a function.

One writes

$$\begin{aligned} f : \mathbb{D} &\rightarrow \mathbb{T} \\ x &\mapsto y = f(x) \end{aligned}$$

The expression $y = f(x)$ is called equation of function.

Definition 2. A function of the form $f : x \mapsto y = mx + t$ with $m, t \in \mathbb{R}$ fixed is said to be linear. m and t are called parameter of the function.

The parameter t gives the distance to the abscisse (x-axis) and the y-intercept is given by $T = (0, t)$. The parameter m gives the slope of the line and we have $\tan \rho = m$, where ρ is the intersection angle.

The zeros of the term $f(x) = 0$ give the x-intercepts.

Two lines $y = f(x)$ and $y = g(x)$ have either no intersection point (if they have the same slope) or one (if they have different slopes. In the latter case one calculates the point of intersection by setting

$$f(x) = g(x),$$

solving this equation for x and then plugging into f or g to solve for y.

A function is invertible if it is injective, meaning that for $x_1 \neq x_2$ we have necessarily $f(x_1) =\neq f(x_2)$. To find the inverese of a function, solve $y = f(x)$ for x and interchange x and y. One obtains the graph of the inverse function by taking the mirror image of the graph of the original function on the diagonal of the first quadrant.

One can connect functions in in the following way:

1. Addition: $f(x) + g(x) = (f + g)(x)$
2. Subtraction: $f(x) - g(x) = (f - g)(x)$
3. Multiplication: $f(x) \cdot g(x) = f \cdot g(x)$
4. Division: $\frac{f(x)}{g(x)} = \frac{f}{g}(x)$ for $g(x) \neq 0$.
5. $f \circ g(x) = f(g(x))$ In general $f \circ g \neq g \circ f$.

Limits

Definition 3. The limit of $f(x)$ as x goes towards x_0

$$\lim_{x \to x_0} f(x) = f(x_0)$$

If f is not defined at the place x_0 the limit is the continuation of f at this point. A limit exists exactly if the function can be continued at this point. One says that f converges to $f(x_0)$ as x goes to x_0. There are right sided and left sided limits. If they coincide, the function can be continued at this point.

Limit Theorems

Assume $n \in \mathbb{N}$, $k \in \mathbb{R}$ and $f(x)$ and $g(x)$ have a limit as $x \to c$.

1. $\lim_{x \to c} k = k$
2. $\lim_{x \to c} x = c$

3. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$
4. $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
5. $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
6. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided $\lim_{x\to c} g(x) \neq 0$.
7. $\lim_{x\to c} f(x)^n = [\lim_{x\to c} f(x)]^n$
8. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$
9. If $f(x) = g(x)$ for all x in an open interval around c except possibly at c itself and if $\lim_{x\to c} g(x)$ exists, then so does $\lim_{x\to c} f(x)$ and the two lomots coincincide.
10. Substitution Theorem: If $f(x)$ is continuous, then $\lim_{x\to c} f(x) = f(c)$ assuming $f(c)$ is defined.
11. Squeeze Theorem: Let f, g, h be functions satisfying $f(x) \leqslant g(x) \leqslant h(x)$ for all x near c, except possibly at $x = c$. If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$.

Trigonometric limits.

The Substitution THeore holds for trigonometric functions on their domains. The following equalities hold:

$$\begin{aligned} \lim_{t\to 0} \frac{\sin t}{t} &= 1 \\ \lim_{7} t \to 0 \frac{1-\cos t}{t} &= 0 \end{aligned}$$

Limits at infinity, infinite limits

Definition 4. Let f be defined on $[c, \infty)$ resp. $(-\infty, c]$ for some $c \in \mathbb{R}$. We say that

$$\lim_{x\to\infty \text{ or } -\infty} f(x) = L$$

if $\forall \epsilon > 0 \exists$ a corresponding number M such that

$$x > M \quad (\text{or } x < M) \Rightarrow |f(x) - L| < \epsilon.$$

Definition 5. We say that

$$\lim_{x\to c^+} f(x) = \infty$$

if $\forall$ positive numbers $M \exists$ a corresponding $\delta > 0$ such that

$$0 < x - c < \delta \Rightarrow f(x) > M.$$

Similar for leftsided limits as well as $-\infty$.

Continuity

Definition 6. A function is contiuous at the point x_0 if for every $\epsilon > 0$ there is $\delta > 0$ such that for $|x - x_0| < \delta$ we hvae $|f(x) - f(x_0)| < \epsilon$.

A function is continuous on the domain, if it is continuous at each point of the domain. One can check continuity with the limit property.

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \widetilde{f}(x_0)$$

Continuous functions:

1. All polynomials are continuous everywhere.
2. All rational functions are continuous where they are defined.
3. Absolute value function is continuous.
4. $f(x) = \sqrt[n]{x}$ is continuous
5. Trigonometric functions are continuous.
6. If f and g are continuous at c then so are
$$kf,\ f+g,\ f-g,\ fg,\ \frac{f}{g}\quad(\text{if } g(c)\neq 0),\ f^n,\ \sqrt[n]{f}.$$
7. If g is contiuous at c and f is continuous at $g(c)$ then $f \circ g$ is continuous at c.

Intermediate Value Theorem

Theorem 7. *Let f be a function defined on $[a,b]$ and $w \in [f(a), f(b)]$. If f is continuous on $[a,b]$ then there exists a number $c \in [a,b]$ such that $f(c) = w$.*