4 The p-typical de Rham–Witt complex

Most of what we say here is taken from Illusie's paper [1]. If X is a smooth $\mathbb{F}_p$-scheme, one could naively try to take the de RHam complex of $W(X)$, and compute the hypercohomology. But it turns out that this doesn't work — it is not even compatible with taking the limit $\lim W_n(\mathscr{O}_X) = W(\mathscr{O}_X)$ (it is not functorial in X). On the other hand the limit of the de Rham complexes of $W_n(X)$ is not compatible with Frobenius and Verschiebung. Thus Deligne's idea was to extend the projective system $W_\bullet(\mathscr{O}_X)$ to a projective system of dga's $W_\bullet \Omega_X)$ and also extend the operators F and V satisfying suitable equalities.

4.1 Definition for $\mathbb{F}_p$-algebras

Following the intuition from the de Rham complex, we will define the de Rham-WItt complex as initial object in a certain category.

Definition 4.1. Let X be a topos. A de Rham-V-procomplex is a projective system

$$M_\bullet = ((M_n)_{n\in\mathbb{Z}}, R : M_{n+1} \to M_n)$$

of $\mathbb{Z}$-dga's on X and a family of additive maps

$$(V : M_n^i \to M_{n+1}^i)_{n\in\mathbb{Z}}$$

such that $RV = VR$ satisfying the following conditions:

(V1) $M_{n\leqslant 0} = 0$, M_1^0 is an $\mathbb{F}_p$-algebra and $M_n^0 = W_n(M_1^0)$ where R and V are the usual maps.

(V2) For $x \in M_n^i$ and $y \in M_n^j$

$$V(xdy) = (Vx)dVy.$$

(V3) For $x \in M_1^0$ and $y \in M_n^0$

$$(Vy)d[x] = V([x]^{p-1}y)dV[x].$$

A morphism of de Rham-V-procomplexes is a morphism of a projective system of dga's $(f_n : M_n \to M'_n)_n$ compatible with all the additional structure in the obvious way ($f_{n+1}V = Vf_n$ and $f_n^0 = W_n(f_1^0)$). Thus the de Rham-V-procomplexes form in a natural way a category denoted by $\mathrm{VDR}(X)$. there is a forgetful functor

$$\mathrm{VDR}(X) \to \mathbb{F}_p\,\mathscr{Alg}(X) \quad , \quad M_\bullet \mapsto M_1^0 \tag{4.1}$$

We can now explain the construction of the de Rham–Witt complex.

Theorem 4.2. *The forgetful functor (4.1) has a left adjoint $A \mapsto W_\bullet\Omega_A$: there is a functorial isomorphism*

$$\mathrm{Hom}_{\mathrm{VDR}(X)}(W_\bullet\Omega_A, M_\bullet) \cong \mathrm{Hom}_{\mathbb{F}_p\,\mathscr{Alg}(X)}(A, M_1^0).$$

For $n \in \mathbb{N}$ the morphism of $\mathbb{Z}$-dga's $\pi_n : \Omega_{W_n(A)} \to W_n\Omega_A$ such that $\pi_n^0 = \mathrm{id}$ is surjective and $\pi : \Omega_A \to W_1\Omega_A$ is an isomorphism.

Proof. The construction is inductive in n. Let $W_n\Omega_A = 0$ for $n \leqslant 0$. Then set $W_1\Omega_A = \Omega_A$. Assume that for fixed $n \geqslant 0$ the system $(R : W_i\Omega_A \to W_{i-1}\Omega_A)_{i\leqslant n}$ and the maps $(V : W_{i-1}\Omega_A \to W_i\Omega_A)_{i\leqslant n}$ are constructed, such that the following conditions are satisfied

$(0)_n$ $RVx = VRx$ for $x \in W_i\Omega_A$, $i \leqslant n-1$.

$(1)_n$ $W_i\Omega_A^0 = W_i(A)$ for $i \leqslant n$ and there V and R are as usual.

$(2)_n$ $V(xdy) = (Vx)dVy$ for $x, y \in W_i\Omega_A$, $i \leqslant n-1$.

$(3)_n$ $(Vy)d[x] = V([x]^{p-1}y)dV[x]$ for $x \in A$, $y \in W_i(A)$, $i \leqslant n-1$.

$(4)_n$ $\pi\Omega_{W_i(A)} \to W_i\Omega_A$ is an epimorphism for $i \leqslant n$.

Now we construct $W_{n+1}\Omega_A$ together with R and V satisfying $(0)_{n+1}, \dots, (4)_{n+1}$.

Let $v : W_n(A)^{\otimes i+1} \to \Omega_{W_{n+1}(A)}$ given by

$$(a \otimes x_1 \otimes \cdots \otimes x_i) \mapsto VadVx_1 \dots dVx_i$$

and $\varepsilon : W_n(A)^{\otimes i+1} \to \Omega^i_{W_n(A)}$ by

$$(a \otimes x_1 \otimes \cdots \otimes x_i) \mapsto adx_1 \dots dx_i$$

Let K^i be the kernel of the composition

$$W_n(A)^{\otimes i+1} \xrightarrow{\varepsilon} \Omega^i_{W_n(A)} \xrightarrow{\pi_n}$$

then $\oplus_i v(K^i)$ is a graded ideal of $\Omega_{W_n(A)}$ (but not stable by d in general). Furthermore, let I be the $W_{n+1}(A)$-submodule of $\Omega^1_{W_{n+1}(A)}$ generated by sections of the form $Vy.d[x] - V([x]^{p-1}y)dV[x]$. Let N be the dgi of $\Omega_{W_{n+1}(A)}$ generated by I and $\oplus_i v(K^i)$. Then we define

$$W_{n+1}\Omega_A := \Omega_{W_{n+1}(A)}/N$$

and π_{n+1} is then just the projection $\Omega_{W_{n+1}(A)} \to W_{n+1}\Omega_A$. The restriction $R : W_{n+1}(A) \to W_n(A)$ induces a morphism of dga's

$$R : \Omega_{W_{n+1}(A)} \to \Omega_{W_n(A)}$$

and because $\pi_n R(N) = 0$ it induces a morphism on the quotients

$$RW_{n+1}\Omega_A \to W_n\Omega_A.$$

Moreover, since by construction $\pi_{n+1}v(K^i) = 0$, V induces an additive map

$$V : W_n\Omega_A \to W_{n+1}\Omega_A$$

satisfying the desired properties. The remaining properties $(0)_{n+1}, \dots, (4)_{n+1}$ are easily verified.

It remains to show that the constructed complex satisfies the desired universal property.

Let $M_\bullet$ be a de Rham-V-procomplex and $f_1^0 : A \to M_1^0$ a homomorphism. Then there is a unique $f_1 : \Omega_A \to M_1$ of dga's extending f_1^0. Inductively, we construct $f_\bullet$.

Assume for $n \geqslant 1$ the morphisms of dga's $f_i : W_i\Omega_A \to M_i$ for $i \leqslant n$ constructed (uniquely because π_i is surjective) such that $f_{i-1}R = Rf_i$, $Vf_{i-1} = f_iV$ and $f_i^0 = W_i(f_1^0)$.

Let $g_{n+1} : \Omega_{W_{n+1}(A)} \to M_{n+1}$ the unique morphism of dga's that extends $W_{n+1}(f_1^0) = f_{n+1}^0$. Then $g_{n+1}(N) = 0$ and the induced map on the quotient $f_{n+1} : W_{n+1}\Omega_A \to M_{n+1}$ satisfies $f_nR = Rf_{n+1}$ and $Vf_n = f_{n+1}V$. The resulting family $f_\bullet$ extends f_1^0 uniquely to a morphism of VDR(X). □

Definition 4.3. Let A be an $\mathbb{F}_p$-algebra of X. The de Rham-V-procomplex $W_\bullet\Omega_A$ is called the de Rham–Witt pro complex of A.

4.2 Some properties

Proposition 4.4. *Let A be as above.*

$$\begin{aligned} xVy &= V(\mathrm{F}\, Rx.y) \quad \text{for } x \in W_n(A), y \in W_{n-1}\Omega^i_A \\ (d[x])Vy &= V(([x]^{p-1}d[x])y) \quad \text{for } x \in A, y \in W_{n-1}\Omega^i_A \end{aligned}$$

Proof. This follows because of the surjectivity directly from (V3) and (V2). □

Proposition 4.5. *Let A be a perfect $\mathbb{F}_p$-algebra. Then $W_\bullet\Omega^i_A = 0$ for $i > 0$.*

Proof. Because of the subjectivity of π it suffices to show this for $\Omega^i_{W_n(A)}$ for $i > 0$ and every n. In fact for a $W_n(A)$-module M any derivation $d : W_n(A) \to M$ is zero: Let $\underline{x} = (x_0, \dots x_{n-1}) \in W_n(A)$. This can be written as the sum $\underline{x} = [x_0] + V[x_1] + \dots + V^{n-1}[x_{n-1}]$, and thus

$$\mathrm{F}^n\, \underline{x} = [x_0]^{p^n} + p[x_1]^{p^{n-1}} + \dots + p^{n-1}[x_{n-1}]^p$$

and $d\,\mathrm{F}^n\,\underline{x}$ is divisible by p^n, and therefore zero. But by hypothesis F is an automorphism (of A), and it follows that d is already zero. □

By construction $W_\bullet\Omega(A)$ is functorial in A, and any morphism of $\mathbb{F}_p$-algebras on X $u : A \to B$ induces a morphism in $\mathrm{VDR}(X)$

$$W_\bullet\Omega_u : W_\bullet\Omega_A \to W_\bullet\Omega_B$$

In particular if k is perfect of characteristic p and A a k-algebra, then $W_n\Omega_A$ is naturally a $W_n(k)$-dga (i.e. d is $W_n(k)$-linear), and V is $\sigma^{-1}W_\bullet(k)$-linear.

Let $k \to k'$ be a morphism of perfect rings of characteristic p and A a k-algebra and $A' = A \otimes k'$, then there is a morphism

$$W_\bullet\Omega_A \otimes W_\bullet(k') \to W_\bullet\Omega_{A'}.$$

Proposition 4.6. *This morphism is an isomorphism.*

Proof. Show this first for the Witt vectors. For this we need that the square

$$\begin{array}{ccc} A' & \xrightarrow{\mathrm{F}} & A' \\ \uparrow & & \uparrow \\ A & \xrightarrow{\mathrm{F}} & A \end{array}$$

is cocartesian, which it is, because k' is perfect. Because we have isomorphisms of dga's

$$\oplus_{n\in\mathbb{N}_0} F^n_* A \xrightarrow{\sim} \mathrm{gr}_V W(A)$$

and similar for A', it follows that for each $n \in \mathbb{N}$

$$W_n(A) \otimes_{W_n(k)} W_n(k') \cong W_n(A')$$

.

Then show that the left hand side is a de Rham-V-procomplex (for this we have to define a Verschiebung:

$$V : W_n\Omega^i_A \otimes W_n(k') \to W_{n+1}\Omega^i_A \otimes W_{n+1}(k') \quad , \quad V(x \otimes FRy) = Vx \otimes y$$

which is the usual V in degree 0). and use universality to extend the identity on A' uniquely to a morphism

$$W_\bullet\Omega_{A'} \to W_\bullet\Omega_A \otimes W_\bullet(k')$$

which is the inverse of the canonical morphism above. □

The functor $W_n(-)$ commutes with inductive filtering limits of $\mathbb{F}_p$-algebras on X. It follows that the category $\mathrm{VDR}(X)$ has filtering inductive limits and if $(A_i)_i$ a filtering inductive system with $A = \varinjlim A_i$, the canonical map

$$\varinjlim W_\bullet\Omega_{A_i} \to W_\bullet\Omega_A$$

is an isomorphism.

In particular, if U is an object of X, the $\Gamma(U, W_\bullet\Omega_A)$ is a de Rham-V-procompelx and

$$W_\bullet\Omega_{\Gamma(U,A)} \to \Gamma(U, W_\bullet\Omega_A)$$

extends the identity in degree zero. This defines a morphism of presheaves which induces an isomorphism on the associated sheaves.

Similar to a statement above, but important in the light of sheaf theory:

Proposition 4.7. *Let $A \to B$ a localisation morphism of $\mathbb{F}_p$-algebras on X (identify B with $S^{-1}A$). Then the $W_\bullet(B)$-linear map*

$$W_\bullet(B) \otimes W_\bullet\Omega^i_A \to W_\bullet\Omega^i_B$$

is an isomorphism

Proof. The idea is similar to above: to show it in degree 0, we need again that the square

$$\begin{array}{ccc} B & \xrightarrow{\mathrm{F}} & B \\ \uparrow & & \uparrow \\ A & \xrightarrow{\mathrm{F}} & A \end{array}$$

is cocartesian (which it is, because we are dealing with a localisation morphism, and $(S^p)^{-1}A = S^{-1}A = B$). Then show that the left hand side is a de Rham-V-procomplex in order to use universality to get an inverse to the morphism in question. □

Now let $(X, \mathscr{O}_X)$ be a ringed tops of $\mathbb{F}_p$-algebras. Then the de Rham–Witt procomplex of $\mathscr{O}_X$ is denoted by

$$W_\bullet \Omega_X.$$

If $f : X \to Y$ is a morphism of ringed topoi of $\mathbb{F}_p$-algebras, then $f_* W_\bullet \Omega_X$ and $f^{-1} W_\bullet \Omega_Y$ are naturally de Rham-V-procomplexes, and there are adjoint maps

$$\begin{array}{rcl} W_\bullet \Omega_Y & \to & f_* W_\bullet \Omega_X \\ f^{-1} W_\bullet \Omega_Y & \to & W_\bullet \Omega_X \end{array}$$

If $\mathscr{O}_X = f^{-1}\mathscr{O}_Y$, the second one is an isomorphism. And in particular, for a point $x \in X$

$$(W_\bullet \Omega_X)_x \to W_\bullet \Omega_{X,x}$$

Proposition 4.8. *For each $n \in \mathbb{N}$ $W_n\Omega^i_X$ is a quasi-coherent sheaf of $W_n(X)$. For each open affine, $U = \operatorname{Spec} A$, we have $\Gamma(U, W_n\Omega^i_X) = W_n\Omega^i_A$.*

Proof. Use the classical methods from basic algebraic geometry. □

Proposition 4.9. *Let $f : X \to Y$ be an étale morphism of $\mathbb{F}_p$-schemes. Then for each n, the $W_n(\mathscr{O}_X)$-linear map*

$$f^* W_n\Omega^i_Y \to W_n\Omega^i_X$$

is an isomorphism.

Proof. It is enough to show this for affine schemes. In this case we have $f : A \to B$ and have to show that

$$W_n(B) \otimes W_n\Omega^i_A \to W_n\Omega^i_B$$

is an isomorphism. For the Witt vectors, we identify again $\operatorname{gr}_V W_n(A)$ with $\oplus_{m<n} F^m_* A$ and similar for B, and we have an isomorphism $B \otimes \operatorname{gr}_V W_n(A) \cong \operatorname{gr}_V W_N(B)$. Moreover, $W_n(f)$ is étale and

$$\begin{array}{ccc} W_n(B) & \xrightarrow{F} & W_n(B) \\ \uparrow & & \uparrow \\ W_n(A) & \xrightarrow{F} & W_n(A) \end{array}$$

is cocartesian.

Because $W_n(B)$ is étale over $W_n(A)$, the derivation of $W_n\Omega_A$ extends uniquely to a derivation on $W_n(B) \otimes W_n\Omega_A$ by

$$d(b \otimes x) = (db)x + b \otimes dx$$

where db is the image of the composition

$$W_n(B) \xrightarrow{d} \Omega^1_{W_n(B)} = W_n(B) \otimes \Omega^1_{W_n(A)} \to W_n(B) \otimes W_n\Omega^1_A.$$

Thus we obtain a projective system of dga's $W_\bullet(B) \otimes W_\bullet \Omega_A$.

To obtain the Verschiebung operator, because the above diagram is cocartesian there is a unique morphism

$$V : W_n(B) \otimes W_n\Omega^i_A \to W_{n+1}(B) \otimes W_{n+1}\Omega^i_A$$

such that $V(FRx \otimes y) = x \otimes Vy$.

This defines a de Rham-V-procomplex and we use universality to get a mao inverse to the original one. □

Definition 4.10. Let X be a ringed topos of $\mathbb{F}_p$-algebras. The complex

$$W\Omega_X := \varprojlim W_n\Omega_X$$

is called the de Rham–Witt complex of X. It is a differential graded algebra, with zero component $W(\mathscr{O}_X)$.

The maps V deine by passing to the limit an additive map V on $W\Omega_X$, which satisfies

$$\begin{aligned} xVy &= V(\mathrm{F}\,x.y) \quad \text{for } x \in W(\mathscr{O}_X), y \in W\Omega^i_X \\ (d[x])Vy &= V(([x]^{p-1}d[x])y) \quad \text{for } x \in \mathscr{O}_X, y \in W\Omega^i_X \\ V(xdy) &= Vx.dVy \quad \text{for } x \in W\Omega^i_X, y \in W\Omega^j_X \end{aligned}$$

4.3 An important example

In order to compare the hyper cohomology of the de Rham–Witt complex with crystalline cohomology, we look first at a basic example. We want to compute the de Rham–Witt complex of $X = (\mathbb{G}^r_a \times \mathbb{G}^s_m)_{\mathbb{F}_p}$. Thus let $A = \mathbb{F}_p[(T_i)_{1\leqslant i\leqslant n}, (T_i^{-1})_{i\in P}]$ where, $n = s+r$ and $P \subset \{1,\dots n\}$, $\#P = s$. (We will in particular need the cases when $s = 0$, i.e. $\mathbb{G}^n_a$, and $s = n$, i.e. $\mathbb{G}^n_m$).

We introduce now the rings

$$\begin{aligned} B &= \mathbb{Z}_p[(T_i)_{1\leqslant i\leqslant n}, (T_i^{-1})_{i\in P}] \\ C &= \bigcup_{r\geqslant 0} \mathbb{Q}_p[(T_i^{p^{-r}})_{1\leqslant i\leqslant n}, (T_i^{-p^{-r}})_{i\in P}] \end{aligned}$$

We have

$$d(T_i^{p^{-r}}) = p^{-r}T_i^{p^{-r}}\frac{dT_i}{T_i}$$

which shows that every form $\omega \in \Omega^m_{C/\mathbb{Q}_p}$ can be written uniquely as

$$\omega = \sum_{i_1<\dots<i_m} a_{i_1\dots i_m}(T)d\log T_{i_1}\dots d\log T_{i_m}$$

with $a_{i_1\dots i_m}(T) \in C$ polynomials over $\mathbb{Q}_p$ in $T_i^{p^{-r}}$ and $T_i^{-p^{-r}}$ for $r \geqslant 0$, divisible by $\prod_{i_j\notin P} T_{i_j}^{p^{-s}}$ for some $s \in \mathbb{N}_0$.

Definition 4.11. We say ω is integral if its coefficients are polynomials over $\mathbb{Z}_p$.

Now we set

$$E^m_A = \left\{\omega \in \Omega^m_{C/\mathbb{Q}_p} \,\middle|\, \omega \text{ and } d\omega \text{ are integral}\right\}$$

which gives a subcomplex $E^\bullet_A \subset \Omega_{C/\mathbb{Q}_p}$ (the biggest subcomplex consisting of integral forms). In particular, it is a sub-dga containing $\Omega_{B/\mathbb{Z}_p}$.

Example 4.12. $T_1^{\frac{1}{p}}$ does not belong to E^0 but $pT_1^{\frac{1}{p}}$ does.

We define two operators F and V on C: an automorphism

$$F(T_i^{p-r}) = T^{p^{-r+1}}$$

and an endomorphism

$$V = pF^{-1}$$

They extend to $\Omega_{C/\mathbb{Q}_p}$ (by acting on the coordinates: $F\sum a_{i_1\dots i_m}(T)d\log T_{i_1}\dots d\log T_{i_m} = \sum Fa_{i_1\dots i_m}(T)d\log T_{i_1}\dots d\log$ and $V\sum a_{i_1\dots i_m}(T)d\log T_{i_1}\dots d\log T_{i_m} = \sum Va_{i_1\dots i_m}(T)d\log T_{i_1}\dots d\log T_{i_m}$), and one verifies

$$dF = pFd\,,\; Vd = pdV$$

so that in particular, $E^\bullet$ is stable by F and V. Furthermore, one has for $x, y \in \Omega_{C/\mathbb{Q}_p}$

$$\begin{aligned} xVy &= V(Fx.y) \\ V(xdy) &= (Vx)(dVy) \end{aligned}$$

The idea now is to set $E_n^m = E^m/(V^nE^m + dV^nE^{m-1})$ and to get a complex

$$\cdot \to E^\bullet_{n+1} \to E^\bullet_n \to E^\bullet_{n-1} \to \cdots$$

The identification $E^0/V^nE^0 \cong W_n(A)$ then induces a structure of V-procomplex $E^\bullet_\bullet$, and we will see that the induced morphism

$$W_\bullet\Omega_A \to E^\bullet_\bullet$$

is in fact an isomorphism.

We will start with the following proposition.

Proposition 4.13. *Keep all the notation from before.*

1. *E^0 is the set of elements $x = \sum a_kT^k \in C$ (using multi indices) such that $a_k \in \mathbb{Z}_p$ and the denominators of all k_i divide a_k.*
2. *We have the identities*

$$\begin{aligned} E^0 &= \sum_{n\in\mathbb{N}_0} V^nB \\ \bigcap_{n\in\mathbb{N}_0} V^nE^0 &= 0 \\ B\cap V^nE^0 &= p^nB \end{aligned}$$

3. *The homomorphism of $\mathbb{Z}_p$-algebras $B \to W(A)$ sending $T_i \mapsto [T_i]$ to its Teichmüller representative, extends in a unique way to a morphism of $\mathbb{Z}_p$-algebras*

$$\tau : E^0 \to W(A)$$

such that $\tau V = V\tau$, It is injective and induces for each $r \in \mathbb{N}$ an isomorphism

$$E^0/V^rE^0 \xrightarrow{\sim} W(A)/V^rW(A).$$

Proof. The first claim follows by definition: x has to be integral, so $a_k \in \mathbb{Z}_p$. For $dx = \sum ka_kT^kd\log T$ to be integral, the $ka_k \in \mathbb{Z}_p$. Note that k_i is of the form $\frac{k_i'}{p^{r_i}}$ with $k_i \in \mathbb{Z}$ and $r_i \in \mathbb{N}_0$, and $(k_i', p^{r_i}) = 1$. Thus the denominator has to divide a_k.

For the second claim, first identity: it is clear that $\sum V^nB \subset E^0$. On the other hand, let $x = aT^k \in E^0$, and p^s the biggest denominator of the k_i. Then we have just seen, that $p^s|a$ and thus we can write $aT^k = V^sp^{-s}aT^{p^sk}$ with $p^{-s}aT^{p^sk} \in B$.

Second and third identity : $x = \sum a_kT^k \in V^nE^0$ means $p^n|a_k$ for all k. Taking the limit over n induces $x = 0$. Also, then $B \cap V^nE^0 = p^nB$ is clear.

For the third claim: Existence of the morphism τ. Set

$$\begin{aligned} \overline{A} &= \bigcup_{r\geqslant 0} \mathbb{F}_p[(T_i^{p^{-r}})_{1\leqslant i\leqslant n}, (T_i^{-p^{-r}})_{i\in P}] \\ \overline{B} &= \bigcup_{r\geqslant 0} \mathbb{Z}_p[(T_i^{p^{-r}})_{1\leqslant i\leqslant n}, (T_i^{-p^{-r}})_{i\in P}] \end{aligned}$$

We have $E^0 \subset \overline{B}$ and F on $\overline{B}$ given by $T_i^{p^{-r}} \mapsto T_i^{p^{-r+1}}$ is an automorphism. Since $\overline{A}$ is perfect, The Witt vector Frobenius on $W(\overline{A})$ is also an automorphism. The morphism of $\mathbb{Z}_p$-algebras

$$\overline{B} \to W(\overline{A}) \ , \ T_i^{p^{-r}} \mapsto [T_i^{p^{-r}}]$$

is compatible with F and therefore with $V = pF^{-1}$. Thus the restriction to $E_0 = \sum_{n\in\mathbb{N}_0} V^nB$ induces the desired morphism τ (as it has image in $W(A)$). It is unique because of the identity $E^0 = \sum_{n\in\mathbb{N}_0} V^nB$.

Now to prove the isomorphism of the quotients mod V^r, note that V^r induces an A-linear homomorphism $F_*^rA \to V^rE^0/V^{r+1}E^0$ and an A-linear iso $F_*^rA \xrightarrow{\sim} V^rW(A)/V^{r+1}W(A)$ and we get a commutative diagram

$$\begin{array}{ccc} & F_*^rA & \\ \swarrow & & \searrow \\ V^rE^0/V^{r+1}E^0 & \xrightarrow{gr_V} & V^rW(A)/V^{r+1}W(A). \end{array}$$

To show that $E^0/V^rE^0 \to W(A)/V^rW(A)$ is an isomorphism, it is enough to show that the horizontal morphism in this diagram gr_V is an isomorphism, hence that $F^r_*A \to V^rE^0/V^{r+1}E^0$ is an isomorphism. Since V is injective on E^0, it is enough to consider $r = 0$, i.e. we have to see that the inclusion $B \subset E^0$ induces an isomorphism $A = B/pB \xrightarrow{\sim} E^0/VE^0$, which follows form the first and third equality of the second claim: $E^0 = \sum_{n\in\mathbb{N}_0} V^nB$ and $B \cap V^nE^0 = p^nB$. Passing to the limit, we obtain an isomorphism

$$\varprojlim E^0/V^rE^0 \xrightarrow{\sim} W(A)$$

and composing with the canonical application $E^0 \to \varprojlim E^0/V^rE^0$ gives exactly τ. And because of the second equality from above, $\bigcap_{n\in\mathbb{N}_0} V^nE^0 = 0$, $E^0 \to \varprojlim E^0/V^rE^0$ is injective, and therefore τ is injective. □

Now we consider the filtration

$$\operatorname{Fil}^r E^i = V^rE^i + dV^rE^{i-1}$$

For each r, the $\operatorname{Fil}^r E^i$, $i \geqslant 1$ form a dgi of $\operatorname{Fil}^r E$ and we have

$$\operatorname{Fil}^0 E = E \supset \operatorname{Fil}^1 E \supset \cdots \supset \operatorname{Fil}^r E \supset \cdots$$

which gives a projective system of dga's

$$E_r = E/\operatorname{Fil}^r E$$

By definition we have $V(\operatorname{Fil}^r E) \subset \operatorname{Fil}^{r+1} E$ and $F\operatorname{Fil}^{r+1} E \subset \operatorname{Fil}^r E$, so that V induces an additive morphism, ad F a morphism of dga's

$$V : E_r \to E_{r+1} \text{ and } F : E_{r+1} \to E_r$$

satisfying the "usual" formulae

$$\begin{cases} dF = pFd, & Vd = pdV \\ xVy = V(Fx.y) & \text{for } x \in E_{r+1}, y \in E_r \\ V(xdy) = Vx.dVy & \text{for } x, y \in E_r \end{cases} \tag{4.2}$$

Theorem 4.14. *The projective system $E_\bullet$ with the operator V and the identification $E^0_r \cong W_r(A)$ for $r \geqslant 1$ is a de Rham-V-procomplex. Moreover, the map*

$$W_\bullet\Omega_A \to E_\bullet$$

extending the identity of A is an isomorphism

In order to prove this, we have to study the structure of E. We will use the notion of basic Witt differentials, which was picked up by Langer and Zink later in their relative construction.

The ring C introduced above has a natural grading, of type

$$G = \left\{ k \in \mathbb{Z}[\frac{1}{p}]^n \mid k_i \geqslant 0 \text{ for } i \notin P \right\}$$

meaning, that the degree of an element is given by the multi-exponents of the variables, which are integers possibly divided by p, negative for $i \in P$, and positive for $i \notin P$ We can extend this grading to $\Omega_{C/\mathbb{Q}_p}$ by saying that a form has degree $k \in G$ if its coordinates are of this degree. Then $E \subset \Omega_{C/\mathbb{Q}_p}$ is a graded sub-complex. Denote the homogeneous component of degree k by ${}_k\Omega_{C/\mathbb{Q}_p}$ and similar or E.

We will use this to find a basis for E. Let $k \in G$ such that $\nu_p(k_1) \leqslant \cdots \leqslant \nu_p(k_n)$. Note that here if k_1 is an integer, so are all k_i, and if $k_r = 0$, then $k_{i\geqslant r} = 0$. Let I_m be the set of integer tuples $(\underline{i} = (i_1, \dots, i_m)$ such that $i_1 \leqslant \cdots \leqslant i_m$ and $k_{i_j} > 0$ for j such that $i_j \notin P$. Then we set

$$t_0 = \begin{cases} 1 & \text{if } i_i = 1 \\ p^{-\nu_p(k_1)} T^k_{[1,i_1[} & \text{if } i_i > 1 \text{ and } k_1 \notin \mathbb{Z}, \\ T^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } k_1 \in \mathbb{Z} \end{cases}$$

and for $s \geqslant 1$

$$t_s = p^{-\nu_p(k_s)} T^k_{[i_s,i_{s+1}[}$$

Then we define

$$e_i(k) = t_0 \prod_{s\geqslant 1, k_{i_s}\neq 0} dt_s \prod_{s\geqslant 1, k_{i_s}=0} d\log T_{i_s} \in_k \Omega^m_{C/\mathbb{Q}_p}$$

and

$$e_0(k) = \begin{cases} p^{-\nu_p(k_1)}T^k & \text{if } k_1 \notin \mathbb{Z}, \\ T^k & \text{otherwise} \end{cases}$$

Proposition 4.15. *Let $k \in G$ such that $\nu_p(k_1) \leqslant \cdots \leqslant \nu_p(k_n)$. For $m \in \mathbb{N}$, the $\mathbb{Z}_p$-module ${}_kE^m$ is free of finite type. The element $e_0(k)$ is a basis for ${}_kE^0$, and for $m \geqslant 1$, the elements $e_{\underline{i}}(k)$ for $\underline{i} \in I_m$ form a basis of ${}_kE^m$.*

Proof. This is a relatively technical proof, that involves juggling around with differentials. It is done by induction. For now I want to omit it. □

The general case, where k does not satisfy $\nu_p(k_1) \leqslant \cdots \leqslant \nu_p(k_n)$, can be deduced from this by applying permutations, as can be imagined easily. More precisely, for each k, we may choose a permutation σ_k, that reorders k, only if the above hypothseis is not satisfied. We denote with a prime the new objects.

Proposition 4.16. *E is generated by E^0 as $\mathbb{Z}_p dga$ (i.e. the $\mathbb{Z}_p$-dga morphism $\Omega_{E^0/\mathbb{Z}_p} \to E$ is surjective), and for each $r \geqslant 1$, Fil^r is a dgi of E generated by V^rE^0.*

Proof. The first claim follows directly after identifying a basis of the homogenous components in the previous proposition: we look at the homogenous components. For the integral components ($k_1 \in \mathbb{Z}$ and therefore all other $k_i \in \mathbb{Z}$) this is just a classical statement. For the case $k - 1 \notin \mathbb{Z}$, note that $de_{\underline{i}}(k) = e_{(1,\underline{i})}(k)$ and these elements generate ${}_kE^{m+1}$ as a $\mathbb{Z}_p$-module.

For the second claim, let I^r_E (or$I^r_{E^0}$) be the dgi generated by V^rE^0 in E (in E^0). Since $\mathrm{Fil}^r E^0 = I^r_{E^0} = V^rE^0$, th inclusion $\mathrm{Fil}^r E \supset I^r_E$ is clear. The other inlcusion follows from the fact, that E^0 generates E as $\mathbb{Z}_p$-algebra. □

We also need to know, what happens to the basic differentials, if we apply the operators V and F as well as the derivative d to them.

Proposition 4.17. *Let $k \in G$ and $k' = (k_{\sigma_k(i)})$ as described previously. For $m \in \mathbb{N}$ and $\underline{i} \in I_m$*

1. *If $1 < i_1$ or $m = 0$*

$$de_{\underline{i}}(k) = \begin{cases} p^{\nu_p(k'_1)}e_{(1,\underline{i})}(k) & \text{if } k'_1 \in \mathbb{Z} \\ e_{(1,\underline{i})}(k) & \text{if } k'_1 \notin \mathbb{Z} \end{cases}$$

If $i_1 = 1$,

$$de_{\underline{i}}(k) = 0$$

2. *If $1 < i_1$ or $m = 0$*

$$Ve_{\underline{i}}(k) = \begin{cases} pe_{\underline{i}}(\frac{k}{p}) & \text{if } \frac{k'_1}{p} \in \mathbb{Z} \\ e_{\underline{i}}(\frac{k}{p}) & \text{if } \frac{k'_1}{p} \notin \mathbb{Z} \end{cases}$$

If $i_1 = 1$,

$$Ve_{\underline{i}}(k) = pe_{\underline{i}}(\frac{k}{p})$$

3. *If $1 < i_1$ or $m = 0$*

$$Fe_{\underline{i}}(k) = \begin{cases} e_{\underline{i}}(pk) & \text{if } k'_1 \in \mathbb{Z} \\ pe_{\underline{i}}(pk) & \text{if } k'_1 \notin \mathbb{Z} \end{cases}$$

If $i_1 = 1$,

$$Fe_{\underline{i}}(k) = e_{\underline{i}}(pk$$

Proof. It is enough to show this for the reordered k. In this case, it just follows from the definition. □

Proposition 4.18. *Let $r \in \mathbb{N}$, $k \in G$. Set $s = s(k) = -\inf_{1 \leqslant i \leqslant n} \nu_p(k_i)$, and*

$$\nu(r,k) = \begin{cases} r - s & \text{if } s > 0, r \geqslant s \\ 0 & \text{if } s > 0, r < s \\ r & \text{if } s \leqslant 0 \end{cases}$$

Then

$${}_k \operatorname{Fil}^r E = p^{\nu(r,k)}({}_k E).$$

Proof. This is a bit tedious, but not hard. □

Corollary 4.19. *Multiplication by p induces a monomorphism $p : E_r \to E_{r+1}$. The components of*

$$\widehat{E} := \varprojlim E_r$$

are p-torsion free and the canonical map $E \to \widehat{E}$ is injective.

Proof. Since the ideal $\operatorname{Fil}^r E$ has a grading with respect to G, we have

$$E_r = \oplus_{k \in G} {}_k E_r.$$

For a chosen homogeneous component one verifies easily, that multiplication by p induces a monomorphism ${}_k E_r \to_k E_{r+1}$. The first claim follows. Hence, it is also true that $\widehat{E}$ is p-torsion free. Moreover, for each $k \in G$, $\bigcap_{r \in \mathbb{N}_0} {}_k \operatorname{Fil}^r E = 0$, so that the canonical map $E \to \widehat{E}$ is injective. □

We are now in a good position to proof the main theorem of this section. For the first part, we have to see, that the system $E_\bullet$ with V and $E^0_r = W_r(A)$ is a de Rham-V-procomplex. Since we have verified the formulae (4.2), the only point to verify form the definition of de Rham-V-procomplex is (V3) $(Vy)d[x] = V([x]^{p-1}y)d[x]$ for $x \in A$ and $y \in E^0_m$. It is sufficient to prove $Fd[x] = [x]^{p-1}d[x]$ because then

$$V([x]^{p-1}y)dV[x] = V([x]^{p-1}ydx) = V(yFd[x]) = d[x].Vy$$

First note, that by passing to the limit $F : E_r \to E_{r-1}$ defines an endomorphism of graded algebras on $\widehat{E}$ such that $dF = pFd$. With $F[x] = [x]^p$ we have $pFd[x] = dF[x] = p[x]^{p-1}d[x]$. As E^1 is p-torsion free, we can divide by p, and get the desired equality.

By the universal property of $W_\bullet\Omega_A$, this means that the identity on A now extends to a morphism of de Rham-V-pro complexes

$$\phi_\bullet : W_\bullet\Omega_A \to E_\bullet$$

and we have to show, that it is in fact an isomorphism. We will construct an inverse to this, by sending the base elements $e_i(k)$ of $E_\bullet$ to certain elements of $W_\bullet\Omega_A$.

We consider again the case $k \in G$ with $\nu_p(k_1) \leqslant v_2 \leqslant \cdots \leqslant \nu_p(k_n)$ — more general cases follow again with permutations. Let $f_0(k) \in W(A)$ be

$$f_0(k) = \begin{cases} p^{-\nu_p(k_1)}[T]^k & \text{if } k_1 \notin \mathbb{Z} \\ [T]^k & \text{if } k_1 \in \mathbb{Z} \end{cases}$$

For $m \geqslant 1$ and $\underline{i} \in I_m$

$$y_0 = \begin{cases} 1 & \text{if } i_1 = 1 \\ p^{-\nu_p(k_1)}[T]^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } k_1 \notin \mathbb{Z} \\ [T]^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } k_1 \in \mathbb{Z} \end{cases}$$

For $s \geqslant 1$ such that $v_p(i_s) < 0$

$$y_s = p^{-\nu_p(k_{i_s})}[T]^k_{[i_s,i_{s+1}[}$$

and for $s \geqslant 1$ such that $0 \leqslant \nu_p(k_{i_s}) < \infty$

$$z_s = [T]^{p^{-\nu_p(k_{i_s})}k}_{[i_s,i_{s+1}[}$$

Now set $f_{\underline{i}}(k) \in W\Omega_A^m$ to be

$$f_{\underline{i}}(k) = y_0 \prod_{s\geqslant 1, \nu_p(k_{i_s})<0} dy_s \prod_{s\geqslant 1, 0\leqslant \nu_p(k_{i_s})<\infty} z_s^{p^{\nu_p(k_{i_s})}-1} dz_s \prod_{s\geqslant 1, \nu_p(k_{i_s})=\infty} d\log[T_{i_s}].$$

Now we define a map $E_\bullet \to W_\bullet \Omega_A$ by sending

$$e_i(k) \mapsto f_i(k)$$

One verifies without difficulty that this commutes with d and V. It is compatible with the filtration on both sides if we define a filtration

$$\mathrm{Fil}'^r W\Omega_A = V^r W\Omega_A + dV^r W\Omega_A^{\bullet -1}$$

which is contained in $\ker(W\Omega_A \to W_r\Omega_A$. Thus, we defined a projective system of morphism of complexes

$$\psi_\bullet E_\bullet \to W_\bullet \Omega_A$$

By definition, $\phi_\bullet \psi_\bullet = \mathrm{id}$, hence it is sufficient, to show that $\psi_\bullet$ is surjective.

Consider the injection $B \subset E^0 \subset W(A)$, which extends to a morphism of $\mathbb{Z}_p$-dga's $\Omega_B \to \Omega_{W(A)}$ which together with the canonical projection gives

$$\Omega_B \to W\Omega_A$$

and this in turn is just the restriction of ψ as they coincide on the base elements $e_i(k)$ for $k \in G \cap \mathbb{Z}^n$.

Let $M \subset W\Omega_A$ be the sub-$\mathbb{Z}_p$-dga generated by $[T]^k$ for $k \in G\cap \mathbb{Z}^n$, $M_\bullet$ its image in $W_\bullet\Omega_A$. Then

$$\psi_\bullet(E_\bullet) \supset M_\bullet$$

Since $\psi_\bullet$ is compatible with V, the subjectivity results form the following identity

$$W_j\Omega_A^i = \sum_{0\leqslant r<j} V^r M_{j-r}^i + \sum_{0\leqslant r<j} dV^r M_{j-r}^{i-1}$$

This need some computation to verify, the interested reader should do it as an exercise.

This finishes the proof of the main theorem.

4.4 The endomorphism F on $W\Omega$

The Frobenius on $E_\bullet$ induces a Frobenius morphism on $W_\bullet\Omega_A$-

Theorem 4.20. *Let X be a ringed topos of $\mathbb{F}_p$-algebras. The homomorphism of projective systems $RF = FR : W_\bullet \mathscr{O}_X \to W_{\bullet -1}\mathscr{O}_X$ extends uniquely to a morphism of projective systems of graded algebras*

$$F : W_\bullet\Omega_X \to W_{\bullet-1}\Omega_X$$

such that for $x \in \mathscr{O}_X$

$$Fd[x] = [x]^{p-1}d[x]$$

and

$$FdV = d : W_n\mathscr{O}_X \to W_n\Omega_X^1$$

In particular, $Fd : W_n\mathscr{O} \to W_{n-1}\Omega_X^1$ is given by the formula

$$Fdx = [x_0]^{p-1}d[x_0] + d[x_1] + \ldots + dV^{n-2}[x_{n-1}]$$

Uniqueness follows from the fact, that an element $x \in W_n\mathscr{O}_X$ can be written as

$$x = [x_0] + V[x_1] + \ldots + V^{n-1}[x_{n-1}]$$

(and from subjectivity of the projection $\Omega_{W_n\mathscr{O}_X} \to W_n\Omega_X$). The uniqueness also implies, that for a morphism of topoi $f : X \to Y$, the induced morphism

$$W_\bullet\Omega_Y \to f_* W_\bullet \Omega_X$$

is compatible with F. We can pass to limits to get a homomorphism of graded algebras

$$F : W\Omega_X \to W\Omega_X$$

satisfying the usual equalities. Note however, that this endomorphism, since it is an endomorphism of complexes, coincides with p^iF in degree i. It would be a useful exercise to show this explicitly.

4.5 Comparison with crystalline cohomology

During this section, let S be a perfect scheme of characteristic $p > 0$ - e.g. $S = \operatorname{Spec} k$ as before. Let $f : X \to S$ be a an S-scheme of finite type. Let $u_n : (X/W_n(S))_{\mathrm{cris}} \to X_{\mathrm{zar}}$ be the canonical projection of topoi. We will define a morphism

$$Ru_n(\mathscr{O}_{X/W_n}) \to W_n\Omega_X \tag{4.3}$$

and show that it is a quasi-isomorphism in case f is smooth. By applying Rf_* and $R\Gamma(X, -)$ to this morphism, one obtains morphisms

$$Rf_{X/W_n}(\mathscr{O}_{X/W_n}) \to Rf_*(W_n\Omega_X)$$

with $f_{X/W_n} = f \circ u_{X/W_n} : (X/W_n)_{\mathrm{cris}} \to (W_n)_{\mathrm{zar}}$, as well as

$$\begin{array}{rcl} R\Gamma_{\mathrm{cris}}(X/W_n) & \to & R\Gamma(X, W_n\Omega) \\ H^\bullet_{\mathrm{cris}}(X/W_n) & \to & H^\bullet(X, W_n\Omega) \end{array}$$

which are also isomorphisms in case X/S is smooth.

Let us start by constructing the morphism (4.3). Assume first, that there is a closed immersion $X \hookrightarrow Y$ in a formal smooth schemes over W endowed with a Frobenius lift $F : Y \to Y^\sigma = Y \times_\sigma W$. For $Y_n = Y \times W_n$ let $\overline{Y}_n$ be the PD-envelope of X in Y_n. In this setup, recall Berthelot's comparison theorem

Theorem 4.21. *There is a canonical quasi-isomorphism*

$$Ru_n(\mathscr{O}_{X/W_n}) \xrightarrow{\sim} \mathscr{O}_{\overline{Y}_n} \otimes \Omega_{Y_n/W_n} = \Omega_{\overline{Y}_n/W_n,[-]}$$

where on the right hand side, we find the PD-de Rham complex.

This sets us up to construct a morphism from the PD-de Rham complex on the right hand side to the de Rham-Witt complex.

From the existence of a Frobenius lift, it follows, that the closed immersion $X \hookrightarrow Y$ extends to an immersion $W_n(X) \hookrightarrow Y$. Namely, let

$$\mathscr{O}_Y \xrightarrow{t_F} W_(\mathscr{O}_{Y_1}) \to i_{1*}W_n(\mathscr{O}_X)$$

where the second arrow is by functoriality given by $i_1 : X \hookrightarrow Y_1$. It sends the ideal $p^n\mathscr{O}_Y$ into $i_{1*}V^nW(\mathscr{O}_X)$ and induces a morphism

$$\mathscr{O}_{Y_n} \to i_{1*}W_n(\mathscr{O}_X). \tag{4.4}$$

Thus, we want to factor $X \to \overline{Y}_n$ through $W_n(X)$. The morphism (4.4) sends the ideal of $X \hookrightarrow Y_n$ to $i_{1*}VW_{n-1}(\mathscr{O}_X)$, which has a natural PD-structure given by

$$\gamma_n(Vx) = \frac{p^{n-1}}{n!}V(x^n)$$

Hence, we can consider the induced PD-morphism

$$\mathscr{O}_{\overline{Y}_n} \to W_n(\mathscr{O}_X).$$

This induces a morphism of de Rham complexes

$$\Omega_{\overline{Y}_n} \to \Omega_{W_n\mathscr{O}_X} \xrightarrow{\pi_n} W_n\Omega_X$$

factoring through the PD-de Rham complex $\Omega_{\overline{Y}_n,[-]} = \Omega_{\overline{Y}_n}/(d\gamma_k(x) = \gamma_{k-1}(x)dx)$.

$$\begin{array}{ccccc} \mathscr{O}_{\overline{Y}_n} \otimes \Omega_{Y_n/W_n} & \longrightarrow & \Omega_{\overline{Y}_n/W_n,[-]} & \longrightarrow & \Omega_{W_n(\mathscr{O}_X)}/(d\gamma_k(Vx) - \gamma_{k-1}(Vx)dx) \\ \uparrow{\scriptstyle\sim} & \searrow & & \searrow & \downarrow \\ Ru_n\mathscr{O}_{X/W_n} & & \longrightarrow & & W_n\Omega_X \end{array}$$

One shows that this construction is independent of choices (of Y and F), by considering for two different Y, Y' with Frobenius lifts F, F' the product $(i, i')X \hookrightarrow Z = Y \times_W Y'$ and $G = F \times_W F'$ to get diagrams

$$\begin{array}{ccccc} Ru_n\,\mathscr{O}_{X/W_n} & \xrightarrow{\sim} & \Omega_{\overline{Y}_n/W_n,[-]} & \longrightarrow & W_n\Omega_X \\ \| & & \downarrow & & \downarrow \\ Ru_n\,\mathscr{O}_{X/W_n} & \xrightarrow{\sim} & \Omega_{\overline{Z}_n/W_n,[-]} & \longrightarrow & W_n\Omega_X \end{array}$$

In general, we can't assume the existence of a closed immersion $X \hookrightarrow Y$ factoring through $W_r(X)$ globally, but only locally. Then one uses a descent argument with respect to an appropriate covering. This will be an exercise.

We come to the main result of this section.

Theorem 4.22. *The morphism (4.3) is a quasi-isomorphism.*

Proof. Because this is a local question, we may assume that X and S are affine – $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} k$ – and choose a flat p-adically complete lift B of A over $W(k)$, together with a Frobenius lift F compatible with σ.

To define the comparison morphism as above, use the immersion of X in the formal scheme $Y = \operatorname{Spf}(B)$ together with F. The ideal of $B_r \to A$ is pB_r, which has a natural PD-structure extending the canonical one. Thus we don't have to modify it to obtain the PD-envelope: $\overline{B}_n = B_n$ and

$$Ru_r\,\mathscr{O}_{X/W_n} \xrightarrow{\sim} \Omega_{B_r}.$$

Using t_F as above, we obtain a morphism $B_n \to W_r(A)$ so

$$\Omega_{B_r} \to W_r\Omega_A,$$

which we have to show is a quasi-isomorphism. It is the same to take the limit on both sides

$$\Omega_B \to W\Omega_A,$$

and show that it induces a quasi-isomorphism on graded pieces for the $p$adic filtration on Ω_B and the canonical filtration on $W\Omega_A$

$$\operatorname{Fil}^r W\Omega_X = \begin{cases} W\Omega_X & \text{if } r \leqslant 0 \\ \ker(W\Omega_X \to W_r\Omega_X) & \text{if } r \geqslant 1 \end{cases}$$

The question is local, so by étale localisation we may reduce to the case, when $A = \mathbb{F}_p[\underline{T}]$, $B = \mathbb{Z}_p[\underline{T}]$ and $C = \mathbb{Q}_p[\underline{T}]$ (to see this, let A be étale over $\mathbb{F}_p[\underline{T}]$, then by functoriality there is an isomorphism $W_rA \otimes \operatorname{Fil}^n W_r\Omega_{\mathbb{F}_p[\underline{T}]} \xrightarrow{\sim} \operatorname{Fil}^n W_r\Omega_A$, so it is enough to consider $A = \mathbb{F}_p[\underline{T}]$).

So we can consider the complex $E^\bullet_\cdot$ defined earlier: we have to show that $\Omega_B/p^n \to E^\bullet_n$ is a quasi-isomorphism. We know that there is an injection

$$\Omega_B \hookrightarrow E^\bullet \hookrightarrow \Omega_{C/\mathbb{Q}_p}$$

Recalling the grading G introduced earlier, we note, that Ω_B consists exactly of thus forms in $E^\bullet$ that have integral weight. Thus we have for each r

$$E^\bullet_r \cong \Omega_{B_r} \oplus \bigoplus_{g \in G, g \notin \mathbb{Z}^n} {}_gE^\bullet_r$$

Delgine showd that for $g \notin \mathbb{Z}^n$ the complex ${}_gE_r$ is homotopically trivial. It follows that the inclusion $\Omega_B \hookrightarrow E$ is a homotopy equivalence, and for each r the inclusion $p^r\Omega_B \hookrightarrow \operatorname{Fil}^r E$ is a homotopy equivalence, such that

$$\Omega_{B_r} = \Omega_B/p^r\Omega_B \hookrightarrow E_r$$

is a quasi-isomorphism. □

It remains to show Deligne's result.

Proposition 4.23. *For $g \notin \mathbb{Z}^n$, the complex ${}_gE$ is homotopically trivial.*

Proof. Wlog we may assume that $g_1 \notin \mathbb{Z}$ (thus $g_1^{-1} \in \mathbb{Z}$). We have to find a homotopy. For this, let h be the operator on $\Omega_{C/\mathbb{Q}_p}$ given by the inner product with $g_1^{-1} T_1 \frac{d}{dT_1}$: for $x = \sum_{i_1 < \ldots < i_m} a_{i_1,\ldots,i_m}(T) d\log T_{i_1} \cdots d\log T_{i_m} \in \Omega_C^m$

$$hx = g_1^{-1} \sum_{i_1 < \ldots < i_m} a_{i_1,\ldots,i_m}(T) d\log T_{i_2} \cdots d\log T_{i_m}.$$

In particular, if x is an integral (i.e. has integral coefficients) form, hx is also integral, and h preserves the weight (homogenous degree) g, which is measured solely on the coefficients. With this definition, the commutator

$$\theta_{g_1^{-1} T_1 \frac{d}{dT_1}} = dh + hd$$

can be seen as the Lie derivative (using the notation of Cartan, nowadays often denoted by $\mathscr{L}_{g_1^{-1} T_1 \frac{d}{dT_1}}$, “Cartan’s magic formula”). Hence, if x is of weight g

$$(dh + hd)(x) = x$$

This is obviously true for function $a(T)$, and because of $d\theta_X \omega = \theta_X d\omega$ with a form ω and a vector field X, this is true in general. Moreover, since by hypothesis dx is integral, hdx is by the above reasoning also integral and so is $dhx = x - hdx$. Thus indeed $hx \in {}_gE$ and h gives a homotopy on ${}_gE$ between the identity and the zero map. □

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