1 Introduction

The de Rham–Witt complex plays an important role in arithmetic geometry, it occurs in different forms and shapes at different places. Historically, the p-typical de Rham-Witt complex as we know it nowadays goes back to Illusie. Why would one mix the concpet of de Rham complex with Witt vectors in the first place? Chambert-Loir in his survey [1] gives (at least) two reasons:

— To have a concrete and intrinsic way to compute crystalline cohomology of a scheme in characterisitic $p \neq 0$ (which is a characteristic 0 object), one would like to have some sort of complex with similar properties as the de Rham complex. The de Rham complex itself does not work, so one need some sort of modification. In particular, it turns out that one needs "divided powers" - which naturally occur in the ring of Witt vectors.

— It was also hoped that such a complex would allow to compare crystalline cohomology to other cohomology theory. Is there for example an analogue of the Hodge to de Rham spectral sequence, can one relate the Serre cohomology $H^*(X, W\mathscr{O})$ or étale cohomology $H^*_{\text{ét}}(X \otimes \overline{k}, \mathbb{Z}_p)$ to crystalline cohomology,?

Bloch was the first who gave a construction a such complex using K-theory in order to answer these questions. However, it was restricted to small enough dimensions and primes $p \neq 2$. Deligne later suggested a construction using differential calculus, which was then carried out by Illusie, and Illusie –Raynaud.

The de Rham–Witt complex as defined by Illusie is a complex of sheaves on a scheme over a perfect field k of characteristic $p \neq 0$. There are generalissations of this to $\mathbb{Z}_{(p)}$-schemes by Langer and Zink [5] – the relative de Rham–Witt complex – and by Hesselholt and Madsen [4] resepctively – the absolute de Rham–WItt complex.

Even further goes the big de Rham–Witt complex due to Hesselholt and Madsen. It is a multi-prime version of the de Rham–Witt complex which is closely related to homological algebra, as it was introduced with the purpose of giving an algebraic description of the equivariant homotopy groups in low degrees of Bökstedt's topological Hochschild spectrum of a commutative ring.This functorial algebraic description, in turn, is essential for understading algebraic K-theory by means of the cyclotomic trace map of Bökstedt–Hsiang–Madsen [3]. There is an improvement of this construction due to Lars Hesselholt using the theory of λ-rings.

If there is interest, it is possible to discuss this more detailed later on in the course.

References

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