Weil restriction

Veronika Ertl

Fakultät für Mathematik
Universität Regesburg

summer term 2020

Idea

k - a commutative ring with one (often a field)

A - a k-algebra

Base change gives us a way to pass from k-schemes to A-scheme:

$$- \otimes_k A : \mathrm{Ens}_k \to \mathrm{Ens}_A, X \mapsto X \times_{\mathrm{Sp}(k)} \mathrm{Sp}(A)$$

where $X \times_{\mathrm{Sp}(k)} \mathrm{Sp}(A)$ is the functor

$$\mathrm{Alg}_k \to \mathrm{Set}, \qquad R \mapsto X(R) \times_{\mathrm{Sp}(k)(R)} \mathrm{Sp}(A)(R)$$

This can be generalised to morphisms of schemes $X \to S$ and base change along $S' \to S$.
Itcan also be extended to non-representable functors in Ens_k.

Question

Is it possible to go the other way round?

Idea

k - a commutative ring with one (often a field)
A - a k-algebra

Base change gives us a way to pass from k-schemes to A-scheme:

$$- \otimes_k A : \mathrm{Ens}_k \to \mathrm{Ens}_A, X \mapsto X \times_{\mathrm{Sp}(k)} \mathrm{Sp}(A)$$

where $X \times_{\mathrm{Sp}(k)} \mathrm{Sp}(A)$ is the functor

$$\mathrm{Alg}_k \to \mathrm{Set}, \qquad R \mapsto X(R) \times_{\mathrm{Sp}(k)(R)} \mathrm{Sp}(A)(R)$$

This can be generalised to morphisms of schemes $X \to S$ and base change along $S' \to S$.
Itcan also be extended to non-representable functors in Ens_k.

Question

Is it possible to go the other way round?

Idea

- k - a commutative ring with one (often a field)
- A - a k-algebra

Base change gives us a way to pass from k-schemes to A-scheme:

$$- \otimes_k A : \mathrm{Ens}_k \to \mathrm{Ens}_A, X \mapsto X \times_{\mathrm{Sp}(k)} \mathrm{Sp}(A)$$

where $X \times_{\mathrm{Sp}(k)} \mathrm{Sp}(A)$ is the functor

$$\mathrm{Alg}_k \to \mathrm{Set}, \qquad R \mapsto X(R) \times_{\mathrm{Sp}(k)(R)} \mathrm{Sp}(A)(R)$$

This can be generalised to morphisms of schemes $X \to S$ and base change along $S' \to S$.
Itcan also be extended to non-representable functors in Ens_k.

Question

Is it possible to go the other way round?

Idea

k - a commutative ring with one (often a field)

A - a k-algebra

Base change gives us a way to pass from k-schemes to A-scheme:

$$- \otimes_k A : \mathrm{Ens}_k \to \mathrm{Ens}_A, X \mapsto X \times_{\mathrm{Sp}(k)} \mathrm{Sp}(A)$$

where $X \times_{\mathrm{Sp}(k)} \mathrm{Sp}(A)$ is the functor

$$\mathrm{Alg}_k \to \mathrm{Set}, \qquad R \mapsto X(R) \times_{\mathrm{Sp}(k)(R)} \mathrm{Sp}(A)(R)$$

This can be generalised to morphisms of schemes $X \to S$ and base change along $S' \to S$.
Itcan also be extended to non-representable functors in Ens_k.

Question

Is it possible to go the other way round?

Definition

Definition

Let $X' \in \mathrm{Ens}_A$.

$$\prod_{A/k}(X') : \mathrm{Alg}_k \to \mathrm{Ens}, R \mapsto X'(A \otimes_k R)$$

Thus $\prod_{A/k} : \mathrm{Ens}_A \to \mathrm{Ens}_k$.

Question

When is $\prod_{A/k}(X') \in Ens_k$ a scheme? More precisely, when is it representable by a k-scheme?

If this is the case, we call $\prod_{A/k} X'$ the Weil restriction of X' from A to k.

Definition

Definition

Let $X' \in \mathrm{Ens}_A$.

$$\prod_{A/k}(X') : \mathrm{Alg}_k \to \mathrm{Ens}, R \mapsto X'(A \otimes_k R)$$

Thus $\prod_{A/k} : \mathrm{Ens}_A \to \mathrm{Ens}_k$.

Question

When is $\prod_{A/k}(X') \in \mathrm{Ens}_k$ a scheme? More precisely, when is it representable by a k-scheme?

If this is the case, we call $\prod_{A/k} X'$ the Weil restriction of X' from A to k.

Definition

Definition

Let $X' \in \mathrm{Ens}_A$.

$$\prod_{A/k}(X') : \mathrm{Alg}_k \to \mathrm{Ens}, R \mapsto X'(A \otimes_k R)$$

Thus $\prod_{A/k} : \mathrm{Ens}_A \to \mathrm{Ens}_k$.

Question

When is $\prod_{A/k}(X') \in Ens_k$ a scheme? More precisely, when is it representable by a k-scheme?

If this is the case, we call $\prod_{A/k} X'$ the Weil restriction of X' from A to k.

Some remarks about the definition

- This definition doesn't use that X' is a scheme - it works for functors.
- We can extend this definition to morphisms $S' \to S$ of k-schemes: Let X' be a contravariant functor $(\mathrm{Sch}/S') \to \mathrm{Ens}$.

$$\prod_{S'/S}(X') : (\mathrm{Sch}/S) \to \mathrm{Ens}, Y \mapsto X'(Y \times_S S')$$

  Thus $\prod_{S'/S} : \mathrm{Fun}_{S'} \to \mathrm{Fun}_S$.
- If X' is a scheme $X'(Y \times_S S') = \mathrm{Hom}_{S'}(Y \times_S S', X')$ and $X'(A \otimes_k R) = \mathrm{Hom}_{\mathrm{Sp}(A)}(\mathrm{Sp}(A) \times_{\mathrm{Sp}(k)} \mathrm{Sp}(R), X')$
- If $X' = \mathrm{Sp}(B)$, then $X'(A \otimes_k R) = \mathrm{Hom}_A(B, A \otimes_k R)$

Adjunction formula

Lemma

Let $X' : (Sch/S')^\circ \to Ens$ be a functor and T an S-scheme. There is a canonical bijection

$$\mathrm{Hom}_S(T, \prod_{S'/S} X') \xrightarrow{\sim} \mathrm{Hom}_{S'}(T \times_S S', X')$$

functorial in T and X'.

Thus $\prod_{S'/S}$ should be the right adjoint to base change (but I got confused about this trying to reconsile this with the affine case).

The adjunction formula has some nice consequences.

Adjunction formula

Lemma

Let $X' : (Sch/S')^\circ \to Ens$ *be a functor and* T *an* S*-scheme. There is a canonical bijection*

$$\mathrm{Hom}_S(T, \prod_{S'/S} X') \xrightarrow{\sim} \mathrm{Hom}_{S'}(T \times_S S', X')$$

functorial in T *and* X'*.*

Thus $\prod_{S'/S}$ should be the right adjoint to base change (but I got confused about this trying to reconsile this with the affine case).

The adjunction formula has some nice consequences.

Adjunction formula

Lemma

Let $X' : (Sch/S')^\circ \to Ens$ *be a functor and* T *an* S*-scheme. There is a canonical bijection*

$$\mathrm{Hom}_S(T, \prod_{S'/S} X') \xrightarrow{\sim} \mathrm{Hom}_{S'}(T \times_S S', X')$$

functorial in T *and* X'*.*

Thus $\prod_{S'/S}$ should be the right adjoint to base change (but I got confused about this trying to reconsile this with the affine case).

The adjunction formula has some nice consequences.

- Let X' be an S'-scheme. If $\prod_{S'/S} X'$ is scheme, then the identity on it gives rise to a functorial morphism

$$\prod_{S'/S} (X') \times_S S' \to X'.$$

- For an S-scheme X, the identity on $X \times_S S'$ gives rise to a functorial morphism

$$X \to \prod_{S'/S} (X \times_S S').$$

- For a functorial morphism $X' \to Y'$ between contravariant functors on (Sch/S') there is a functorial morphisms

$$\prod_{S'/S} (X') \to \prod_{S'/S} (Y').$$

- Let X' be an S'-scheme. If $\prod_{S'/S} X'$ is scheme, then the identity on it gives rise to a functorial morphism

$$\prod_{S'/S}(X') \times_S S' \to X'.$$

- For an S-scheme X, the identity on $X \times_S S'$ gives rise to a functorial morphism

$$X \to \prod_{S'/S}(X \times_S S').$$

- For a functorial morphism $X' \to Y'$ between contravariant functors on (Sch/S') there is a functorial morphisms

$$\prod_{S'/S}(X') \to \prod_{S'/S}(Y').$$

- Let X' be an S'-scheme. If $\prod_{S'/S} X'$ is scheme, then the identity on it gives rise to a functorial morphism

$$\prod_{S'/S} (X') \times_S S' \to X'.$$

- For an S-scheme X, the identity on $X \times_S S'$ gives rise to a functorial morphism

$$X \to \prod_{S'/S} (X \times_S S').$$

- For a functorial morphism $X' \to Y'$ between contravariant functors on (Sch/S') there is a functorial morphisms

$$\prod_{S'/S} (X') \to \prod_{S'/S} (Y').$$

- $\prod_{S'/S}$ commutes with fibre products
  - $\Rightarrow$ preserves group functors - which is interesting for us in the context of group schemes
  - $\Rightarrow$ it is compatible with base change: $T \to S$ morphism of base change, $T' := S' \times_S T$, X' an S'-scheme

$$\prod_{T'/T}(X' \times_{S'} T') \cong \prod_{S'/S}(X') \times_S T$$

- If X' is a sheaf (for the Zariski topology), the same is true for $\prod_{S'/S}(X')$.

Representability

We already talked about a criterion when for an S'-scheme X' the functor $\prod_{S'/S}(X')$ is an S-scheme.

Theorem

Let $S' \to S$ be finite locally free, and X' an S'-scheme.
***Condition**: For each $s \in S$ and finite set of points $P \subset X' \otimes_S k(s)$ there is an affine open subscheme $P \subset U' \subset X'$.*
Then $\prod_{S'/S}(X')$ is representable by an S-scheme X.

1. By localising, we may assume that $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ are affine, where R' is a free R-module with generators $e_1, \ldots, e_n$.
2. We treat first the affine case.

Let X' be affine.
$\Rightarrow$ It can be seen as a closed subscheme of $\operatorname{Spec} R'[\underline{t}]$ where $\underline{t}$ is a system of vairables (maybe infinite).
$\Rightarrow$ Since $\prod_{S'/S}$ preserves in this situation closed immersions (of functors) it suffices to consider the case $X' = \operatorname{Spec} R'[\underline{t}]$.

1. By localising, we may assume that $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ are affine, where R' is a free R-module with generators $e_1, \ldots, e_n$.
2. We treat first the affine case.

Let X' be affine.

$\Rightarrow$ It can be seen as a closed subscheme of $\operatorname{Spec} R'[\underline{t}]$ where $\underline{t}$ is a system of vairables (maybe infinite).

$\Rightarrow$ Since $\prod_{S'/S}$ preserves in this situation closed immersions (of functors) it suffices to consider the case $X' = \operatorname{Spec} R'[\underline{t}]$.

Consider n-copies of the system $\underline{t}$: $\underline{t}_1, \dots, \underline{t}_n$. We argue that Spec $R[\underline{t}_1, \dots, \underline{t}_n]$ represents $\prod_{S'/S}(X')$. Thus for any R-algebra A we want to define a bijection

$$\mathrm{Hom}_{R'}(R'[\underline{t}], A \otimes_R R') \to \mathrm{Hom}_R(R[\underline{t}_1, \dots, \underline{t}_n], A)$$

which is functorial in A.

Let $\sigma' : R'[\underline{t}] \to A \otimes_R R'$ on the LHS. This is determined by the image of $\underline{t}$ in

$$A \otimes_R R' = \bigoplus_{i=1}^{n} (A \otimes_R R) e_i.$$

Consider n-copies of the system $\underline{t}$: $\underline{t}_1, \ldots, \underline{t}_n$. We argue that Spec $R[\underline{t}_1, \ldots, \underline{t}_n]$ represents $\prod_{S'/S}(X')$. Thus for any R-algebra A we want to define a bijection

$$\mathrm{Hom}_{R'}(R'[\underline{t}], A \otimes_R R') \to \mathrm{Hom}_R(R[\underline{t}_1, \ldots, \underline{t}_n], A)$$

which is functorial in A.

Let $\sigma' : R'[\underline{t}] \to A \otimes_R R'$ on the LHS. This is determined by the image of $\underline{t}$ in

$$A \otimes_R R' = \bigoplus_{i=1}^{n} (A \otimes_R R) e_i.$$

Thus we can write

$$\sigma'(\underline{t}) = \sum_{i=1}^{n} \alpha_i \otimes e_i$$

where coefficients are systems of elements in A.

$\Rightarrow$ This determines a homomorphism

$$\sigma : R[\underline{t}_1, \ldots, \underline{t}_n] \to A, \qquad \underline{t}_i \mapsto \alpha_i.$$

And gives us the desired bijection.

Thus we can write

$$\sigma'(\underline{t}) = \sum_{i=1}^{n} \alpha_i \otimes e_i$$

where coefficients are systems of elements in A.

$\Rightarrow$This determines a homomorphism

$$\sigma : R[\underline{t}_1, \ldots, \underline{t}_n] \to A, \qquad \underline{t}_i \mapsto \alpha_i.$$

And gives us the desired bijection.

3. Next we come to the case when X' is not necessarily affine.

We know already that locally the $\prod_{S'/S}(X')$ is representable. More precisely:

Let $\{U_i'\}_i$ be the system of all affine open subschemes of X'.
$\Rightarrow$ the $\prod_{S'/S}(U_i')$ are representable by affine schemes U_i. In this sitution $\prod_{S'/S}$ preserves open immersions (of functors), thus

$$U_i \hookrightarrow \prod_{S'/S}(X')$$

is an open immersion.

$\Rightarrow$The gluing data of the U_i' as open subschemes of X' gives rise to gluing data for the U_i and hence we obtain an S-scheme X.

$\Rightarrow$ Since X' is in particular a sheaf, the same is true for $\prod_{S'/S}(X')$ and we obtain a functorial morphism

$$X \to \prod_{S'/S}(X').$$

3. Next we come to the case when X' is not necessarily affine.

We know already that locally the $\prod_{S'/S}(X')$ is representable. More precisely:

Let $\{U_i'\}_i$ be the system of all affine open subschemes of X'.

$\Rightarrow$ the $\prod_{S'/S}(U_i')$ are representable by affine schemes U_i. In this sitution $\prod_{S'/S}$ preserves open immersions (of functors), thus

$$U_i \hookrightarrow \prod_{S'/S}(X')$$

is an open immersion.

$\Rightarrow$The gluing data of the U_i' as open subschemes of X' gives rise to gluing data for the U_i and hence we obtain an S-scheme X.

$\Rightarrow$ Since X' is in particular a sheaf, the same is true for $\prod_{S'/S}(X')$ and we obtain a functorial morphism

$$X \to \prod_{S'/S}(X').$$

3. Next we come to the case when X' is not necessarily affine.

We know already that locally the $\prod_{S'/S}(X')$ is representable. More precisely:

Let $\{U_i'\}_i$ be the system of all affine open subschemes of X'.
$\Rightarrow$ the $\prod_{S'/S}(U_i')$ are representable by affine schemes U_i. In this sitution $\prod_{S'/S}$ preserves open immersions (of functors), thus

$$U_i \hookrightarrow \prod_{S'/S}(X')$$

is an open immersion.

$\Rightarrow$The gluing data of the U_i' as open subschemes of X' gives rise to gluing data for the U_i and hence we obtain an S-scheme X.

$\Rightarrow$ Since X' is in particular a sheaf, the same is true for $\prod_{S'/S}(X')$ and we obtain a functorial morphism

$$X \to \prod_{S'/S}(X').$$

3. Next we come to the case when X' is not necessarily affine.

We know already that locally the $\prod_{S'/S}(X')$ is representable. More precisely:

Let $\{U_i'\}_i$ be the system of all affine open subschemes of X'.

$\Rightarrow$ the $\prod_{S'/S}(U_i')$ are representable by affine schemes U_i. In this sitution $\prod_{S'/S}$ preserves open immersions (of functors), thus

$$U_i \hookrightarrow \prod_{S'/S}(X')$$

is an open immersion.

$\Rightarrow$The gluing data of the U_i' as open subschemes of X' gives rise to gluing data for the U_i and hence we obtain an S-scheme X.

$\Rightarrow$ Since X' is in particular a sheaf, the same is true for $\prod_{S'/S}(X')$ and we obtain a functorial morphism

$$X \to \prod_{S'/S}(X').$$

3. Next we come to the case when X' is not necessarily affine.

We know already that locally the $\prod_{S'/S}(X')$ is representable. More precisely:

Let $\{U'_i\}_i$ be the system of all affine open subschemes of X'.
$\Rightarrow$ the $\prod_{S'/S}(U'_i)$ are representable by affine schemes U_i. In this sitution $\prod_{S'/S}$ preserves open immersions (of functors), thus

$$U_i \hookrightarrow \prod_{S'/S}(X')$$

is an open immersion.

$\Rightarrow$The gluing data of the U'_i as open subschemes of X' gives rise to gluing data for the U_i and hence we obtain an S-scheme X.

$\Rightarrow$ Since X' is in particular a sheaf, the same is true for $\prod_{S'/S}(X')$ and we obtain a functorial morphism

$$X \to \prod_{S'/S}(X').$$

To show that $X \to \prod_{S'/S}(X')$ is an equivalence we will use the condition.

It suffices to show that for any S-scheme T each functorial morphism $\alpha : T \to \prod_{S'/S}(X')$ factors uniquely through Y.
$\Rightarrow$ It suffices to show this locally in a neighbourhood of each point $z \in T$.

Let (z_j) be the finite family of points in $T \times_S S'$ above z and

$$\alpha' : T \times_S S' \to X'$$

the morphism corresponding to α. Set $x_j = \alpha'(z_j)$.

By the condition: there is an affine open $U' \subset X'$ containing all points x_j.
As before: $\prod_{S'/S}(U')$ is representable by an S-scheme U and $U \to \prod_{S'/S}(X')$ is an open immersion.

To show that $X \to \prod_{S'/S}(X')$ is an equivalence we will use the condition.

It suffices to show that for any S-scheme T each functorial morphism $\alpha : T \to \prod_{S'/S}(X')$ factors uniquely through Y.
$\Rightarrow$ It suffices to show this locally in a neighbourhood of each point $z \in T$.

Let (z_j) be the finite family of points in $T \times_S S'$ above z and

$$\alpha' : T \times_S S' \to X'$$

the morphism corresponding to α. Set $x_j = \alpha'(z_j)$.

By the condition: there is an affine open $U' \subset X'$ containing all points x_j.
As before: $\prod_{S'/S}(U')$ is representable by an S-scheme U and $U \to \prod_{S'/S}(X')$ is an open immersion.

To show that $X \to \prod_{S'/S}(X')$ is an equivalence we will use the condition.

It suffices to show that for any S-scheme T each functorial morphism $\alpha : T \to \prod_{S'/S}(X')$ factors uniquely through Y.
$\Rightarrow$ It suffices to show this locally in a neighbourhood of each point $z \in T$.

Let (z_j) be the finite family of points in $T \times_S S'$ above z and

$$\alpha' : T \times_S S' \to X'$$

the morphism corresponding to α. Set $x_j = \alpha'(z_j)$.

By the condition: there is an affine open $U' \subset X'$ containing all points x_j.
As before: $\prod_{S'/S}(U')$ is representable by an S-scheme U and $U \to \prod_{S'/S}(X')$ is an open immersion.

To show that $X \to \prod_{S'/S}(X')$ is an equivalence we will use the condition.

It suffices to show that for any S-scheme T each functorial morphism $\alpha : T \to \prod_{S'/S}(X')$ factors uniquely through Y.
$\Rightarrow$ It suffices to show this locally in a neighbourhood of each point $z \in T$.

Let (z_j) be the finite family of points in $T \times_S S'$ above z and

$$\alpha' : T \times_S S' \to X'$$

the morphism corresponding to α. Set $x_j = \alpha'(z_j)$.

By the condition: there is an affine open $U' \subset X'$ containing all points x_j.
As before: $\prod_{S'/S}(U')$ is representable by an S-scheme U and $U \to \prod_{S'/S}(X')$ is an open immersion.

By definition of Y this factors

$$U \to Y \to \prod_{S'/S}(X').$$

Replacing T by a suitable open neighbourhood of z, we may asum that there is a factorisation

$$\alpha' : T \times_S S' \to U' \to X'$$

$\Rightarrow$ then $\alpha : T \to \prod_{S'/S}(X')$ factors through U and hence through Y.

The factorisation is unique as $Y \to \prod_{S'/S}(X')$ is an open immersion. This finishes the proof.

By definition of Y this factors

$$U \to Y \to \prod_{S'/S}(X').$$

Replacing T by a suitable open neighbourhood of z, we may asum that there is a factorisation

$$\alpha' : T \times_S S' \to U' \to X'$$

$\Rightarrow$ then $\alpha : T \to \prod_{S'/S}(X')$ factors through U and hence through Y.

The factorisation is unique as $Y \to \prod_{S'/S}(X')$ is an open immersion. This finishes the proof.

By definition of Y this factors

$$U \to Y \to \prod_{S'/S}(X').$$

Replacing T by a suitable open neighbourhood of z, we may asum that there is a factorisation

$$\alpha' : T \times_S S' \to U' \to X'$$

$\Rightarrow$ then $\alpha : T \to \prod_{S'/S}(X')$ factors through U and hence through Y.

The factorisation is unique as $Y \to \prod_{S'/S}(X')$ is an open immersion. This finishes the proof.

By definition of Y this factors

$$U \to Y \to \prod_{S'/S}(X').$$

Replacing T by a suitable open neighbourhood of z, we may asum that there is a factorisation

$$\alpha' : T \times_S S' \to U' \to X'$$

$\Rightarrow$ then $\alpha : T \to \prod_{S'/S}(X')$ factors through U and hence through Y.

The factorisation is unique as $Y \to \prod_{S'/S}(X')$ is an open immersion. This finishes the proof.

Properties that carry over

Let $S' \to S$ be finite and locally free and X' an S' scheme. If $\prod_{S'/S}(X')$ is an S-scheme, some properties carry over from X'/S' to $\prod_{S'/S}(X')/S$:

1. separated
2. locally of finite type
3. locally of finite presentation
4. finite presentation
5. smooth

If $S' \to S$ is étale:

1. quasi-compact
2. proper
3. flat

If S is locally Noetherian:

1. quasi-compact

Properties that carry over

Let $S' \to S$ be finite and locally free and X' an S' scheme. If $\prod_{S'/S}(X')$ is an S-scheme, some properties carry over from X'/S' to $\prod_{S'/S}(X')/S$:

1. separated
2. locally of finite type
3. locally of finite presentation
4. finite presentation
5. smooth

If $S' \to S$ is étale:

1. quasi-compact
2. proper
3. flat

If S is locally Noetherian:

1. quasi-compact

Properties that carry over

Let $S' \to S$ be finite and locally free and X' an S' scheme. If $\prod_{S'/S}(X')$ is an S-scheme, some properties carry over from X'/S' to $\prod_{S'/S}(X')/S$:

1. separated
2. locally of finite type
3. locally of finite presentation
4. finite presentation
5. smooth

If $S' \to S$ is étale:

1. quasi-compact
2. proper
3. flat

If S is locally Noetherian:

1. quasi-compact

Properties that carry over

Let $S' \to S$ be finite and locally free and X' an S' scheme. If $\prod_{S'/S}(X')$ is an S-scheme, some properties carry over from X'/S' to $\prod_{S'/S}(X')/S$:

1. separated
2. locally of finite type
3. locally of finite presentation
4. finite presentation
5. smooth

If $S' \to S$ is étale:

1. quasi-compact
2. proper
3. flat

If S is locally Noetherian:

1. quasi-compact