Universität Regensburg Fakultät für Mathematik

Lecture notes

Algebraic groups and group schemes

(an inquiry based learning course)

Summer term 2020

Preface

These notes represent our attempt to create a course on Algebraic Groups based on the Moore method. As Aristotle pithily remarks

> "For the things we have to learn before we can do, we learn by doing ."
> –Aristotle: Nicomachean Ethics, Book II – (350 b.c.e)

The hope is that the reader will gain good working knowledge of Algebraic Groups, by actively working through the necessary steps, in contrast to passivley absorbing the theory.

While we picked up this idea out of necessity when non-virtual classes were suspended during the 2019-20 coronavirus pandemic, we think that it presents an excellent way for the participants and readers not only to gain an active knowledge of the topic at hand but also to introduce them to mathematical research and scientific (or other) collaboration.

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Introduction

What is an algebraic group? Some might contend that the name algebraic group is poorly chosen as they are not the groups that one meets as a student in algebra. Rather it is a mathematical object with two structures. This is a phenomena that is not rare in mathematics. Consider for example topological groups, objects with both an algebraic structure and a topological structure.

Roughly speaking, an algebraic group is the algebraic analogue of a Lie group with algebraic geometry playing the role of differential geometry. While a Lie group is simultaneously a differentiable manifold and a group such that the group operations are compatible with the manifold structure, an algebraic group is simultaneously a variety and a group, where the group operations correspond to morphisms. In a categorical language, a topological group is a group object in the category of topological spaces, a Lie group is a group object in the category of smooth manifolds — and an algebraic group is a group object in the category of k-varieties.

To be precise, by an algebraic group G over a field k we mean a connected smooth k-group scheme. In particular, G is separated, of finite type and geometrically integral over k.

The most important classes of algebraic groups are the affine and the projective algebraic groups. It is well-known that the former are exactly the Zariski-closed algebraic subgroups of the matrix groups GL_n and hence are commonly referred to as *linear algebraic groups*. It is useful to think of an affine variety G over k as a functor that assigns to any field extension k'/k the set $G(k')$ of common solutions over k' of some fixed family of polynomials with coefficients in k. For the general linear group GL_n as the basic example, $\mathrm{GL}_n(k')$ is the group of invertible $n \times n$-matrices with entries in k' or the collection of solutions (t, X), $t \in k'$ and $X \in M_{n\times n}(k')$, to the polynomial equation $t \cdot \det X = 1$.

The latter ones are automatically abelian, and hence are called abelian varieties. In dimension one these are exactly the elliptic curves. The term elliptic comes from the term elliptic functions, whose natural domains are Riemann surfaces - elliptic curves in complex geometry. By the end of the 19th century geometric methods were used to study abelian functions. Weil gave the subject its modern foundations in the language of algebraic geometry. Nowadays, abelian varieties provide important tools in number theory and algebraic geometry, but also in the study of dynamical systems.

However other types of algebraic groups arise naturally. For example, for a proper scheme X over a perfect field k, the reduced connected component $(\mathrm{Pic}^0_{X/k})_{\mathrm{red}}$ of its relative Picard scheme is an algebraic group which is in general not proper and not affine over k. If X is of dimensions 1 the arising algebraic group is called generalised Jacobian in geometric class field theory.

An other example is the following: let V be a discrete valuation ring with fraction field K and residue field k and A and abelian variety over K. The connected component of the closed fibre of its Néron model $\mathcal{A}$ over V (the "best possible" group schem defined over V corresponding to A) is an algebraic group over k which is in general not proper and not affine. In fact, properness of the connected component of the closed fibre is equivalent to good reduction.

Accordingly, it is important to understand the general structure of algebraic groups, and this is one goal of this course. One of the main results that we want to understand is Chevalley's theorem, sometimes also referred to as Barsotti–Chevalley theorem.. It is a structure theorem states that every algebraic group over a perfect field is an extension of an abelian variety by a smooth affine algebraic group. In that sense, it measures how far a general algebraic group is from being affine or projective. As a consequence any algebraic group over a field is quasi-projective.

This result has important applications. One is the proof of the Néron–Ogg–Shafarevich criterion for good reduction of abelian varieties. Geometric class field theory, the extension of class field theory to higher geometric objects, which is part of the geometric Langlands program, also depends on Chevalley's theorem.

Since the time of the early proofs of Chevalley's theorem by Chevalley, Barsotti and Rosenlicht, the mathematical language has evolved considerably, and now there are modern versions of the proof available, by Conrad and Milne.

In the course we will lay the ground work to understand Chevalley's theorem and hopefully go beyond that. We welcome input from the participants and readers and update the notes accordingly.

1 A functorial approach to schemes

In this chapter we will set up some basic language.

1.1 Notation and conventions

For a category $\mathscr{C}$ and objects $A, B \in \mathscr{C}$ we denote by $\mathrm{Hom}_{\mathscr{C}}(A, B)$ the class of morphisms form A to B. We denote by $\mathrm{Fon}(\mathscr{A}, \mathscr{B})$ the category of functors from one category $\mathscr{A}$ to another category $\mathscr{B}$.

Notation 1.1.1. Categories that we consider frequently are the category of Sets Ens, the category of groups Gr, the category of abelian groups Ab. Furthermore, for a (commutative) ring (with 1) denote by Alg_k the category of (finitely generated) k-algebras. The category of k-sets Ens_k is the category of functors $\mathrm{Fon}(\mathrm{Alg}_k, \mathrm{Ens})$, and the category of k-groups Gr_k is the category of functors $\mathrm{Fon}(\mathrm{Alg}_k, \mathrm{Gr})$.

Examples 1.1.2. The functors

$$\begin{aligned}
&\mathbb{G}_a : \mathrm{Alg}_k \to \mathrm{Gr}, R \mapsto R, \\
&\mathbb{G}_m : \mathrm{Alg}_k \to \mathrm{Gr}, R \mapsto R^{\times}, \\
&\mu_l : \mathrm{Alg}_k \to \mathrm{Gr}, R \mapsto \{x \in R \mid x^l = 1\}, \\
&\mathbb{A}^n : \mathrm{Alg}_k \to \mathrm{Ens}, R \mapsto R^n,
\end{aligned}$$

are ftypical examples of elements in Gr_k and Ens_k.

Definition 1.1.3. For a functor $F : \mathrm{Alg}_k \to \mathrm{Ens}$ and $R \in \mathrm{Alg}_k$ an R-point of F is an element $\alpha \in F(R)$.

1.2 Affine k-schemes

1.2.1 Representable functors

The basis to consider schemes as functors is the Yoneda lemma, more precisely the following consequence or special case of the Yoneda lemma.

Theorem 1.2.1 (Yoneda lemma)**.** *Let $\mathscr{C}$ be a locally small category (i.e. the hom-classes $\mathrm{Hom}_{\mathscr{C}}(A, B)$ are sets and not only classes). The functor*

$$\begin{aligned}
\mathrm{Sp}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} &\to \mathrm{Fon}(\mathscr{C}, \mathrm{Ens}), \\
A &\mapsto \mathrm{Sp}_{\mathscr{C}}(A) = \mathrm{Hom}_{\mathscr{C}}(A, -)
\end{aligned}$$

is fully faithful.

In other words, for $A, B \in \mathscr{C}$ one has

$$\mathrm{Hom}_{\mathrm{Fon}(\mathscr{C},\mathrm{Ens})}(\mathrm{Sp}_{\mathscr{C}}(A), \mathrm{Sp}_{\mathscr{C}}(B)) \cong \mathrm{Hom}_{\mathscr{C}}(B, A) = \mathrm{Sp}_{\mathscr{C}}(B)(A).$$

Remark 1.2.2. (i) The functor $\mathrm{Sp}_{\mathscr{C}}$ is often denote by h^-, so that $\mathrm{Sp}_{\mathscr{C}}(A) = h^A$ and called the Yoneda embedding.

(ii) The contravariant version is deduced easily from this version.

(iii) More generally, the Yoneda lemma states that for a functore $F : \mathscr{C} \to \mathrm{Ens}$ one has

$$\mathrm{Hom}_{\mathrm{Fon}(\mathscr{C},\mathrm{Ens})}(\mathrm{Sp}_{\mathscr{C}}(A), F) \cong F(A).$$

Definition 1.2.3. We say a functor $F : \mathscr{C} \to \mathrm{Ens}$ is representable if there is $C \in \mathscr{C}$ such that F is ismorphic to $\mathrm{Sp}_{\mathscr{C}}(C)$, in other words, if F is in the essential image of Sp.

Examples 1.2.4. Give your favourit example of a representable functor and the representing object.
We will collect the examples of all participants.

Example 1.2.5. Find an example of a non-representable functor and show that it is non-representable.
A very rough criterion: Let $F : \mathrm{Alg}_k \to \mathrm{Ens}$ be a functor and $A \to B$ an injective morphism of k-algebras such that $F(A) \to F(B)$ is not injective. Then F is not representable.

Definition 1.2.6. A functor $F : \mathrm{Alg}_k \to \mathrm{Ens}$ is an affine k-scheme, if it is representable by a k-algebra A.

In other words, an affine k-scheme is isomorphic to $\mathrm{Sp}_k(A)$ for a k-algebra A.

Theorem 1.2.7. *The category of affine k-schemes has fibre products.*

Proof. Consider the fibre product in the category of sets Ens.
How can we describe the category in the category of k-sets Ens_k?
What object represents the fibre product of two affine k-schemes over a third affine k-scheme? □

1.2.2 The Zariski topology

One can consider an affine k-scheme as a topological space. Let $A \in \mathrm{Alg}_k$.

Recall that $\mathrm{Spec}(A)$ is the set of prime ideals of A. For $\mathfrak{P} \in \mathrm{Spec}(A)$, denote by $\kappa(\mathfrak{P})$ the residue field at $\mathfrak{P}$. For a ring homomorphism $\phi : A \to B$, the preimage of every prime ideal $\mathfrak{P}_B$ in B, $\mathfrak{P}_A = \phi^{-1}(\mathfrak{P}_B)$ is a prime ideal in A. Thus ϕ induces a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

For $f \in A$ let $U_f = \{\mathfrak{P} \in \mathrm{Spec}(A) \mid f \notin \mathfrak{P}\}$ be the subset of $\mathrm{Spec}(A)$ of prime ideals not containing f. These are called the basic open set and $U_f = \mathrm{Spec}(A[f^{-1}])$.

The Zariski topology on $\mathrm{Spec}(A)$ is generated by the U_f (i.e. the Zariski topology is the coarsest topology such that the U_f are open sets).

Open sets in the Zariski topology are of the form

$$U_I = \{\mathfrak{P} \in \mathrm{Spec}(A) \mid I \not\subset \mathfrak{P}\}$$

where $I \subset A$ is an ideal (possibly equal to A). For ideals $I, J \subset A$, one has

$$U_I \cap U_J = U_{IJ}$$
$$U_I \cup U_J = U_{I+J}$$

For a (possibly infinite) family $\{f_\alpha \in A\}_{\alpha \in \mathscr{I}}$

$$\bigcup_{f_\alpha} U_{f_\alpha} = U_I, \qquad \text{where } I \text{ is the ideal generated by the } f_\alpha,$$
$$U_{f_\alpha} \cap U_{f_\beta} = U_{f_\alpha f_\beta}$$

By definition, there is an equivalent of partition of unity: for a family as above such that $(f_\alpha \in A)_{\alpha \in \mathscr{I}} = A$, there is a finite subset $\{f_1, \ldots, f_n\} \subset \{f_\alpha \in A\}_{\alpha \in \mathscr{I}}$ and $g_1, \ldots, g_n \in A$ such that $\sum_{i=1}^n f_i g_i = 1$.

The topological space $\mathrm{Spec}(A)$ is quasi-compact: every open cover of $\mathrm{Spec}(A)$ as a finite subcover.

With this it is possible to identify the underlying topological space of $\mathrm{Sp}_k(A)$. For this we need the following definitions.

Definition 1.2.8. A subfunctor of a functor $F : \mathrm{Alg}_k \to \mathrm{Ens}$ is a functor $F' : \mathrm{Alg}_k \to \mathrm{Ens}$ such that for any $A \in \mathrm{Alg}_k$, $F'(A)$ is a subset of $F(A)$.

Definition 1.2.9. For $f \in A$, define a subfunctor of $\mathrm{Sp}_k(A)$ that corresponds to the basic open U_f in $\mathrm{Spec}(A)$.
Hint: U_f is the subset of $\mathrm{Spec}(A)$ such that every morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ which factors through U_f is induced by a morphism $A \to B$ which factors through $A[f^{-1}]$.
Use these subfunctors to define a topology on $\mathrm{Sp}_k(A)$, or in other words, the underlying topological space $|\mathrm{Sp}_k(A)| = \mathrm{Spec}(A)$.

Definition 1.2.10. Extend this idea to define the open subfunctors of $\mathrm{Sp}_k(A)$?

1.2.3 Locality

In the next section we will globalise these definitions. Thus we have to be able to "glue".

Lemma 1.2.11. *Let $A \in \mathrm{Alg}_k$ and $f_1, \ldots, f_n \in A$ such that $\bigcup_{i=1}^n U_{f_i} = \mathrm{Spec}(A)$. Set $A_i = A[f_i^{-1}]$ and $A_{ij} = A[f_i^{-1}, f_j^{-1}]$. Let $B \in \mathrm{Alg}_k$ and $\beta_i \in \mathrm{Hom}_k(B, A_i)$ such that β_i and β_j have the same image in $\mathrm{Hom}_k(B, A_{ij})$. There is a unique element $\beta \in \mathrm{Hom}_k(B, A)$ with image β_i in $\mathrm{Hom}_k(B, A_i)$.*

Proof. Show this lemma. $\square$

In other words, elements $\beta_i \in \mathrm{Hom}_{\mathrm{Ens}_k}(\mathrm{Sp}_k(A_i), \mathrm{Sp}_k(B))$ such that all β_i, β_j restrict to the same element in $\mathrm{Sp}_k(A_{ij})$, glue to a unique element

$$\beta \in \mathrm{Hom}_{\mathrm{Ens}_k}(\mathrm{Sp}_k(A), \mathrm{Sp}_k(B)).$$

Definition 1.2.12. A functor $F : \mathrm{Alg}_k \to \mathrm{Ens}$ is said to be local if for any $A \in \mathrm{Alg}_k$ and generators $f_1, \ldots, f_n \in A$ of the unit ideal, the first arrow in the diagram

$$\mathrm{Hom}(\mathrm{Sp}_k(A), F) \to \prod_{i=1}^{n} \mathrm{Hom}(\mathrm{Sp}_k(A_i), F) \rightrightarrows \prod_{i,j} \mathrm{Hom}(\mathrm{Sp}_k(A_{ij}), F)$$

is an equaliser.

We have shown that for any $B \in \mathrm{Alg}_k$ the functor $\mathrm{Sp}_k(B)$ is local!

Example 1.2.13. Find a functor which is not local

1.3 k-schemes

1.3.1 Gluing affine k-schemes

We can now define general k-schemes in terms of functors. Recall the definitions 1.2.8 and 1.2.10.

Definition 1.3.1. Based on this, how would you define an open subfunctor for a functor $F : \mathrm{Alg}_k \to \mathrm{Ens}$?

Remark 1.3.2. What is the underlying topological space associated to an open subfunctor of an affine k-scheme? (Maybe we will move this to the previous section later...)

Definition 1.3.3. A functor $X : \mathrm{Alg}_k \to \mathrm{Ens}$ is a k-scheme, if:

- it is local,
- there exist open subfunctors $\{U_i \mid i \in I\}$ of X, which are affine schemes, such that for any $\alpha : \mathrm{Sp}_k(A) \to X$ the open subsets $|\mathrm{Sp}_k(A) \times_X U_i \subset \mathrm{Sp}_k(A)|$ cover the underlying topological space of $\mathrm{Sp}_k(A)$.

We say that the U_i cover X.

As in the affine case, we can consider the underlying topological space of a k-scheme X.

Definition 1.3.4. How can we define the underlying topological space $|X|$?
Hint: consider a suitable equivalence relation on a disjoint union $\sqcup \mathrm{Spec}(A_i)$ where $U_i = \mathrm{Spec}(A_i)$ are as in the previous definition.

In particular, for every point $x \in |X|$ one of the $\mathrm{Spec}(A_i)$ is an open neighbourhood, and hence x corresponds to a prime ideal $\mathfrak{P}_x \in \mathrm{Spec}(A_i)$, and one can consider the associated local ring $(A_i)_{\mathfrak{P}_x}$ and the residue field $\kappa(x)$.

Definition 1.3.5. Let X, Y be k-schemes. A morphism of k-schemes $f : X \to Y$ is a morphism in $\mathrm{Ens}_k = \mathrm{Fon}(\mathrm{Alg}_k, \mathrm{Ens})$. The category of k-schemes is a full subcategory of the category Ens_k which contains the category of affine schemes.

1.3.2 Sheaves

For a k-scheme X, Aff_X is the category with

- objects $A \in \mathrm{Alg}_k$ endowed with a morphism $\mathrm{Sp}_k(A) \to X$ and
- morphisms $B \to A$ such that the diagram

$$\begin{tikzcd} \mathrm{Sp}_k(A) \arrow[rr] \arrow[dr] & & \mathrm{Sp}_k(B) \arrow[dl] \\ & X & \end{tikzcd}$$

  commutes.

Definition 1.3.6. A sheaf on X is a local functor $F : \mathrm{Aff}_X^{\mathrm{op}} \to \mathrm{Ens}$. In other words, for any open affine covering $\coprod_{i \in \mathscr{I}} \mathrm{Sp}_k(A_i) \to \mathrm{Sp}_k(A)$ over X, the sequence

$$F(A) \to \prod_i F(A_i) \rightrightarrows \prod_{i,j} F(A_{ij})$$

with $A_{ij} = A_i \otimes_A A_j$ is an equalizer diagram. We denote by $\mathrm{Shv}(X)$ the full subcategory of sheaves on Aff_X.

Example 1.3.7. Define the structure sheaf $\mathscr{O}_X$ as basic example

Lemma 1.3.8. *For a k-scheme Y over X, every local functor F on Aff_X extends to Y.*

Proof. Show this lemma. □

Example 1.3.9. In particular $\Gamma(X, \mathscr{O}_X)$ is a ring and we can interpret X as a locally ringed space.

1.3.3 Schemes of finite type and separated k-schemes

Definition 1.3.10. A morphism $i : Z \to X$ of k-schemes is a closed immersion, if there exist open affine subfunctors $\{U_j = \mathrm{Sp}_k(A_j) \mid j \in J\}$ which cover X, and for every $j \in J$ an ideal $I_j \subset A_j$ such that $i^{-1}(U_j) = \mathrm{Sp}_k(A_j/I_j)$ as schemes over U_j.

Remark 1.3.11. It turns out that this is equivalent to the following more calssical definition:
A morphism $i : Z \to X$ of k-schemes is a closed immersion, if

- it induces a homeomorphism of $|Z|$ with a closed subset of $|X|$,

- the induced map $i^\sharp : \mathscr{O}_X \to i_*\mathscr{O}_Z$ is surjective.

Yet another characterization, closed immersions are subfunctors factoring isomorphically through a closed subscheme.

For completeness, the following is included.

Definition 1.3.12. Give a definition (in "our" language) of an open immersion.

Remark 1.3.13. Again, this is equivalent to the more classical definition:
A morphism $j : U \to X$ of k-schemes is a open immersion, if

- it induces a homeomorphism of $|U|$ with an open subset of $|X|$,
- the induced map $j^{-1}\mathscr{O}_X \to \mathscr{O}_Z$ is an isomorphism.

Definition 1.3.14. A k-scheme X is of finite type, if it can be covered by finitely many open affine subfunctors $\mathrm{Sp}_k(A_i)$ such that the A_i finitely presented k-algebras.

Definition 1.3.15. A k-scheme is separated, if the diagonal morphisms

$$\Delta_X : X \to X \times_k X$$

is a closed immersion.

Lemma 1.3.16. *Let X be a separated k-scheme. For any two morphisms $a : \mathrm{Sp}_k(A) \to X$ and $b : \mathrm{Sp}_k(B) \to X$, the product $\mathrm{Sp}_k(A) \times_X \mathrm{Sp}_k(B)$ is again an affine scheme.*

Proof. Show this lemma. □

1.3.4 Modules

We define the notion of $\mathscr{O}_X$-modules.

Definition 1.3.17. Let A be a (commutative) ring and M and A-module of finite type. The local rank of M at a prime ideal $\mathfrak{p}$ is defined as

$$\mathrm{rg}_{\mathfrak{p}}(M) = \dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p})).$$

Lemma 1.3.18. *Let M and A be as above. The map*

$$\mathrm{rg}(M) : \mathrm{Spec}(A) \to \mathbb{N}$$

is upper semi-continuous. It is locally constant, if and only if M is locally free.

Lemma 1.3.19. *Let M be an A-module of finite type. The functor*

$$\mathrm{Alg}_A \to \mathrm{Ens}, B \mapsto \mathrm{Hom}_A(M, B)$$

is representable.

Proof. Prove this.
What is a good candidate for a representing object? How can we "make M into an A-algebra $S_A(M)$? ☐

We thus obtain the affine space over A associated to an A-module M.

Corollary 1.3.20. *Let M and A be as before. If M is locally free then the functor*

$$\mathrm{Alg}_A \to \mathrm{Ens}\,, B \mapsto M \otimes_A B$$

is representable,

Proof. Prove this - think of the dual of M. ☐

Remark 1.3.21. The converse of the previous statement is also true. This is a bit trickier to show. You can show this as a bonus.

Definition 1.3.22. Let X be a k-scheme. Define $\mathscr{O}_X$ modules as a functor on $\mathrm{Aff}_X \to \mathrm{Ens}$.

Remark 1.3.23. We can then define the sheaf of algebras $\mathrm{Sym}_{\mathscr{O}_X}(M)$.

1.4 Projective space

We consider the following definition of projective space.

Definition 1.4.1. The functor

$$\mathbb{P}^n : \mathrm{Alg}_k \to \mathrm{Ens}$$

which associates to a k-algebra A the set of isomorphism classes of surjective morphisms $A^{n+1} \to L$ where L is a locally free A-module of rank 1 is called projective space of dimension n.

Remark 1.4.2. There are several characterisations of projective space.
What is your favourite definition?

Lemma 1.4.3. *This functor defines a k-scheme.*

Proof. Give a proof of this.
Note that the canonical basis of A^{n+1} defines morphisms $x_i : A \to A^{n+1}$. This allows us to define subfunctors U_i of $\mathbb{P}^n$ by only considering those surjections $A^{n+1} \to L$, such that composing with x_i gives an isomorphism $A \to L$.
Next we have to show that the U_i are open subfunctors. For $A \in \mathrm{Alg}_k$ let $A^{n+1} \to L$ be an A-point of $\mathbb{P}^n$. We may assume L is free (why?).
How can we describe $|U_i| \cap \mathrm{Spec}(A)$? This should be affine...
Next we have to show that the U_i cover $\mathbb{P}^n$. For this we show that for any A the $|U_i| \cap \mathrm{Spec}(A)$ cover $\mathrm{Spec}(A)$.
Finally, show that $\mathbb{P}^n$ is local. ☐

Remark 1.4.4. Note that if we demanded that L is free, the functor would not be local anymore.

Lemma 1.4.5. *$\mathbb{P}^n$ is endowed with an invertible module with $n+1$ sections.*

Proof. Define the twisting sheaf.
Note that each A-point (for a $A \in \mathrm{Alg}_k$) comes with a module and sections... □

The above definition is only a special case of the projective fibre associated to a module.

Definition 1.4.6. Let $A \in \mathrm{Alg}_k$ and M a locally free A-module of rank $n+1$.
In analogy to the definition above, define the projective fibre $\mathbb{P}_M$ associated to M.

The projective space functor $\mathbb{P}^n$ has the following extension property:

Lemma 1.4.7. *Let $R \in \mathrm{Alg}_k$ be a discrete valuation ring with fraction field K. Every K-point x_K extends in a unique way to an R-point of $\mathbb{P}^n$.*

Proof. Note that to a K-point one can associate an exact sequence of vector spaces. □

Definition 1.4.8. (i) Define what it means for a k-scheme X of finite type to be proper.

(ii) Define what it means for a morphism of k-schemes to be proper.

Corollary 1.4.9. *The functor $\mathbb{P}^n$ is proper (and so is $\mathbb{P}_M$ for a locally free k-module of finite type).*

1.5 Base change and Weil restriction

The category of k-schemes has fibre products.

Example 1.5.1. For $X, Y, Z \in \mathrm{Ens}_k$ which are k-schemes, with morphisms $f : X \to Z$, $g : Y \to Z$, describe the fibre product of X, Y over Z.

In particular we may consider base change:

Definition 1.5.2. Let $A \in \mathrm{Alg}_k$ and $X \in \mathrm{Ens}_k$ a k-scheme.
What is the functor X restricted to the category Alg_A?
We often denote it by $X_A = X \otimes_k A$.

Now we want to consider the inverse of this.

Definition 1.5.3. Let A be a k-algebra, $X \in \mathrm{Ens}_A$ a A-scheme. Consider the functor

$$\prod_{A/k} X : \mathrm{Alg}_k \to \mathrm{Ens}, \quad R \mapsto X(A \otimes_k R).$$

We call the functor $\prod_{A/k}$ Weil restriction.

Remark 1.5.4. By definition, Weil restriction is right adjoint to base change. Let Y be a k-scheme and X an A-scheme, then

$$\mathrm{Hom}_A(Y_A, X) \cong \mathrm{Hom}_k(Y, \prod_{A/k} X).$$

We want to study when the functor $\prod_{A/k} X \in \mathrm{Ens}_k$ is (representable by) a scheme.

Example 1.5.5. Let X be a k-scheme and $A = k \times k$.
Describe $\prod_{A/k}(X \otimes A)$.

Example 1.5.6. Let $A \in \mathrm{Alg}_k$ be finitely generated and free of rank d as a k-module. Let $X = \mathbb{A}^n_A$.
Show that $\prod_{A/k} X$ is representable by an affine space of dimension nd.

Example 1.5.7. Can you find an example where the Weil restriction of a scheme is not representable?

More generally, one can say the following:

Lemma 1.5.8. *Let $A \in \mathrm{Alg}_k$ be finite locally free. Let $X \in \mathrm{Ens}_A$ be an affine A-scheme of finite type. Then $\prod_{A/k} X$ is representable by a k-scheme of finite type.*

Proof. Prove this.
For an A-algebra R describe the R-points of X starting from an algebra that represent X. Use this to describe for a k-algebra B the B-points of $\prod_{A/k} X$. Find the representing affine k-scheme (or the representing k-algebra). □

A nice example is the tangent space.

Example 1.5.9. Let $A = k[\epsilon]$ with $\epsilon^2 = 0$.
Show that for a k-scheme X, $TX := \prod_{A/k}(X \otimes_k A)$ is the total tangent space of X. Consider the morphism of functors defined by $k[\epsilon] \to k$, $\epsilon \mapsto 0$. The goal is to describe the fibres over a point x of X. For this, reduce the question to the affine case $X = \mathrm{Sp}_k(R)$. Then we can consider the corresponding morphism $R \to \kappa(x)$. How can we describe the fibre over x in terms of morphisms?

Weil restriction commutes with base change in the following sense.

Lemma 1.5.10. *Let $A \in \mathrm{Alg}_k$, $X \in \mathrm{Ens}_A$. Moreover, let k' be a k-algebra and $A' = A \otimes_k k'$*

$$\prod_{A/k} X \otimes_k k' = \prod_{A'/k'} X',$$

where $X' = X \otimes_A A'$.

Proof. Give a proof for this. □

Example 1.5.11. Let k be an algebraically closed field, $A = k[u]$, $B = k[v]$. Consider the k-algebra morphism $\phi : A \to B$, $u \mapsto v^2$. Thus a morphism $\mathrm{Sp}_k(B) \to \mathrm{Sp}_k(A)$. Le $\mathbb{G}_m$ be the grou functor over B and $G := \prod_{B/A} \mathbb{G}_m$ the Weil restriction of $\mathbb{G}_m$ from B to A. This is an affine scheme over A over A.
Can you describe the fibres of G over $\mathrm{Sp}_k(A)$?

To finish this section we give a criterion for representability in the non-affine case.

Proposition 1.5.12. *Let A be a finite locally free k-algebra and X an algebraic A-scheme, such that for every point $s \in \mathrm{Sp}(k)$ for every finite set of points P in $X \times_{\mathrm{Sp}(k)} s$ there is an open affine U of X which contains P. Then $\prod_{A/k} X$ is representable by a k-scheme.*

Proof. Give a proof for this.
Start with an open cover of X. What is the most obvious candidate for an open cover of $\prod_{A/k} X$? You can use that $\prod_{A/k}$ sends open subfunctors to open subfunctors. □

1.6 Flatness

1.6.1 Faithful and flat morphisms

Recall the definition of flatness: Let A be a (commutative) ring (with 1). For every A-module M, the tensor functor $- \otimes_A M$ is right exact. The A-module M is called flat, if this functor is also left exact. An A-algebra B is called flat, if it is flat as an A-module.

A free A-module of finite rank is trivially flat. An inductive limit of flat modules is flat. In fact, every flat A-module is an inductive limit of free A-modules of finite rank.

Example 1.6.1. Give your favourite example of a flat and a non-flat module.

Definition 1.6.2. A morphism of k-schemes $f : Y \to X$ is flat if there are open affine coverings $V = \sqcup \mathrm{Sp}_k(B_\beta) \to Y$ and $U = \sqcup \mathrm{Sp}_k(A_\alpha) \to X$, and a map of indices $\beta \mapsto \alpha$ such that f restricted to $\mathrm{Sp}_k(B_\beta)$ induces a morphism $\mathrm{Sp}_k(B_\beta) \to \mathrm{Sp}_k(A_\alpha)$ which come from flat morphisms of k-algebras $A_\alpha \to B_\beta$.

Remark 1.6.3. We will se that a flat morphism of k-schemes $f : Y \to X$ which is locally of finite presentation induces an open map on the underlying topological spaces $|f| : |Y| \to |X|$. (Note that if X is locally Noetherian, f being locally of finite type implies f being locally of finite presentation.)

Recall that an A-module M is faithfully flat, if the tensor functor $- \otimes_A M$ is exact and faithful. An A-algebra B is faithfully flat, if it is faithfully flat as A-module

Remark 1.6.4. (i) For a faithfully flat A-algebra B, show that if M is a non-trivial A-module, the $M \otimes_A B$ is a non-trivial B-module.

(ii) For a faithfully flat A-algebra B, show that a sequence of A-modules $0 \to M \to N \to P \to 0$ is exact if and only if the sequence of B-modules $0 \to M \otimes_A B \to N \otimes_A B \to P \otimes_A B \to 0$ is exact.

Lemma 1.6.5. *A flat A-algebra B is faithfully flat if and only if the induced morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective.*

Proof. Give a proof for this. □

Remark 1.6.6. We leaf the general case to the reader.

Definition 1.6.7. From the above lemma deduce a sensible definition for a faithfully flat morphism of schemes.

In the next chapter, going-up and going down will play an important role. We recall here some notions and a statement for flat morphisms.

Remark 1.6.8. Recall that for $x, x' \in X$ with $x \in \overline{\{x'\}}$ we say that x' is a generalisation of x and x is a specialisation of x'.

(i) One says that a subset $Y \subset X$ is is stable under specialisation if for every $x' \in T$ and specialisation $x \in \overline{\{x'\}}$ one has $x \in Y$.

(ii) One says that a subset $Y \subset X$ is is stable under generalisation if for every $x \in T$ and generalisationx' of x, i.e. $x \in \overline{\{x'\}}$, one has $x' \in Y$.

(iii) Let $X = \mathrm{Spec} A$. Let $x, x' \in X$ corresponding to $\mathfrak{p}$ and $\mathfrak{p}'$. Then

$$x \in \overline{\{x'\}} \Leftrightarrow \mathfrak{p}' \subset \mathfrak{p}.$$

(iv) A ring morphism $\varphi : A \to B$ satisfies going down if for prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ in A and $\mathfrak{q}'$ in B such that $\mathfrak{q}' \cap A = \mathfrak{p}'$, there is a prime $\mathfrak{q} \subset \mathfrak{q}'$ with $\mathfrak{q} \cap A = \mathfrak{p}$.

(v) A ring morphism $\varphi : A \to B$ satisfies going up if for prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ in A and $\mathfrak{q}$ in B such that $\mathfrak{q} \cap A = \mathfrak{p}$, there is a prime $\mathfrak{q} \subset \mathfrak{q}'$ with $\mathfrak{q}' \cap A = \mathfrak{p}'$.

(vi) Let $f : Y \to X$ be a morphism of schemes. We say that generalisatoins lift along f if for any $x \in \overline{\{x'\}}$ in X such that there is $y \in Y$ with $f(y) = x$, there is y' with $y \in \overline{\{y'\}}$ in Y such that $f(y') = x'$. If Y and X are spectra of rings this is equivalent to going down.

(vii) Let $f : Y \to X$ be a morphism of schemes. We say that specialisations lift along f if for any $x \in \overline{\{x'\}}$ in X such that there is $y' \in Y$ with $f(y') = x'$, there is $y \in \overline{\{y'\}}$ in Y such that $f(y) = x$. If Y and X are spectra of rings this is equivalent to going up.

Lemma 1.6.9. *A flat morphism $\varphi : A \to B$ of rings satisfies going-down.*

Proof. Let $\mathfrak{p} \subset \mathfrak{p}'$ be prime ideals of A and $\mathfrak{q}'$ a prime ideal in B such that $\mathfrak{q}' \cap A = \mathfrak{p}'$. The induced map of local rings $A_{\mathfrak{p}'} \to B_{\mathfrak{q}'}$ is also flat and in particular faithfully flat. By the previous lemma the associated map on spectra is surjective and hence one finds a prime in $B_{\mathfrak{q}'}$ mapping to $\mathfrak{p}A_{\mathfrak{p}'}$. Thus let $\mathfrak{q}$ be the preimage in B of this prime and we are done. □

1.6.2 Faithfully flat descent

Let $f\colon A \to B$ be a morphism of rings. Then canonical diagram of rings

$$A \longrightarrow B \rightrightarrows B \otimes_A B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} B \otimes_A B \otimes_A B$$

induces a diagram of categories of modules

$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B \rightrightarrows \mathrm{Mod}_{B\otimes_A B} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathrm{Mod}_{B\otimes_A B\otimes_A B}$$

by extending scalars.

Definition 1.6.10. The category of descent data with respect to $f\colon A \to B$ is given by the homotopy limit of the diagram

$$\mathrm{holim}\,(\mathrm{Mod}_B \rightrightarrows \mathrm{Mod}_{B\otimes_A B} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathrm{Mod}_{B\otimes_A B\otimes_A B})$$

This is given in the following way:

- An object (N, φ) is a B-module N together with an isomorphism

  $$\varphi\colon N \otimes_A B \to B \otimes_A N$$

  of $B \otimes_A B$-modules. Such that the triangle

  $$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[rr, "\sigma_{B,B}\circ(\mathrm{id}\otimes\varphi)\circ\sigma_{N,B}"] \arrow[dr, "\varphi\otimes\mathrm{id}"'] & & B \otimes_A B \otimes_A N \\ & B \otimes_A N \otimes_A B \arrow[ur, "\mathrm{id}\otimes\varphi"'] & \end{tikzcd}$$

  commutes. Here we abused notation slightly since $\sigma_{B,B} \circ (\mathrm{id} \otimes \varphi) \circ \sigma_{N,B}$ actually should be denoted $(\sigma_{B,B} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) \circ (\sigma_{N,B} \otimes \mathrm{id})$, if σ is the braiding.

- A morphism $(N, \varphi) \to (M, \psi)$ is a morphism $f\colon N \to M$ of B-modules, such that the square

  $$\begin{tikzcd} N \otimes_A B \arrow[r, "f\otimes\mathrm{id}"] \arrow[d, "\varphi"'] & M \otimes_A B \arrow[d, "\psi"] \\ B \otimes_A N \arrow[r, "\mathrm{id}\otimes f"'] & B \otimes_A M \end{tikzcd}$$

  commutes.

Definition 1.6.11. We say, that a morphism $f\colon A \to B$ satisfies descent, if the canonical functor

$$\mathrm{Mod}_A \longrightarrow \mathrm{holim}\,(\mathrm{Mod}_B \rightrightarrows \mathrm{Mod}_{B\otimes_A B} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathrm{Mod}_{B\otimes_A B\otimes_A B})$$

is an equivalence of categories.

Definition 1.6.12. Let $\mathcal{C}$ be a category. Then a *split equalizer* is a diagram of the form

$$\begin{tikzcd} C \arrow[r, "h"] & D \arrow[l, bend left, "s"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & E \arrow[l, bend left, "t"] \end{tikzcd}$$

such that $f \circ h = g \circ h$, $s \circ h = \mathrm{id}_C$, $t \circ g = \mathrm{id}_D$ and $t \circ f = h \circ s$.

Remark 1.6.13. We leave it as an easy exercise, to see, that split equalizers are equalizers that are preserved by any functor.

Lemma 1.6.14. *Let A be as above and A' a faithfully flat A-module. Consider the tensor product $A'' := A' \otimes_A A'$ with the obvious double arrow $A' \rightrightarrows A''$ given by $\mathrm{pr}_1 : a' \mapsto a' \otimes 1$ and $\mathrm{pr}_2 : a' \mapsto 1 \otimes a$. Then for any A-module M there is an exact sequence*

$$0 \to M \to M' \rightrightarrows M''$$

where $M' = M \otimes_A A'$, $M'' = M \otimes_A A$" and the morphisms are induced from the corresponding algebra morphisms.

Proof. Sketch the proof. Use base change by A' and item (3) of the remark above. Note that

$$0 \to A' \to A'' \rightrightarrows A'''$$

is exact, and argue why tensoring with M' preserves exactness in this case. □

Remark 1.6.15. Classically, descent can be formulated in the following terms:

Let A' be a faithfully flat A-algebra and N an A'-module. A descent datum for N with respect to $A \to A'$ is an isomorphism

$$\rho \in \mathrm{Hom}_{A''}(\mathrm{pr}_1^* N, \mathrm{pr}_2^* N)$$

such that the diagram

$$\begin{tikzcd} \mathrm{pr}_1^*\mathrm{pr}_1^* N \arrow[rr, "\rho_{12}"] & & \mathrm{pr}_1^*\mathrm{pr}_2^* N \arrow[dl, "\rho_{23}"] \\ & \mathrm{pr}_2^*\mathrm{pr}_2^* N \arrow[ul, "\rho_{31}"] & \end{tikzcd}$$

with $\rho_{12} = \rho \otimes \mathrm{id}$, etc. commutes.

Proposition 1.6.16. *Let A' be a faithfully flat A-algebra. Then the morphism of rings*

$$A \to A'$$

satisfies descent.

Proof. Give a sketch of the proof. □

Corollary 1.6.17. *There is an equivalence of categories from the category of A-algebras and the category of A'-algebras with A-descent datum.*

Corollary 1.6.18 (Zariski descent)**.** *Let A be a ring and $f_1, \ldots, f_n \in A$ elements generating the ring. Then, if we set $A_i = A[f_i^{-1}]$, the canonical morphism of rings*

$$A \to \prod_{i=0}^{n} A_i$$

satisfies descent. Add finiteness condition for A?

Proof. Give a prof for this. □

Remark 1.6.19. Characterise the category of Zariski descent data.

We want to pass to schemes now.

Definition 1.6.20. A morphism of k-schemes $S' \to S$ is quasi-compact, if for any k-algebra A with morphism $\mathrm{Sp}_k(A) \to S$ the product $S' \times_S \mathrm{Sp}_k(A)$ can be covered by finitely many open affines.

Faithfully flat descent extends to morphisms of schemes $S' \to S$ which are faithfully flat and quasi-compact.

Proposition 1.6.21. *Let $S' \to S$ be faithfully flat and quasi-compact, set $S'' = S' \times_S S'$. For S-schemes X and Y, denote by X', Y' and X'', Y'' the base change by S' and S'', respectively. There is an exact sequence (of sets)*

$$\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_{S'}(X', Y') \rightrightarrows \mathrm{Hom}_{S''}(X'', Y'').$$

Proof. Argue that it is enough to consider S, S' affine. Then reduce to the case when X and Y are aso affine, in which case it follows from the discussion above. □

The above statement can be reformulated as follows: Let X be a k-scheme.
Recall how to see X as a Zariski sheaf on Aff_k.
Because of the above result it is natural to consider a finer topology than the Zariski topology in this context.

Definition 1.6.22. An fpqc-covering is a family $\{\varphi_i : T_i \to T\}$ of k-schemes such that

(i) the φ_i are flat morphisms

(ii) for each open affine subscheme $U \subset T$, there are finitely many open affine subschemes $(U_i \subset T_i)_{i \in J}$ such that $\bigsqcup_{i \in J} \varphi_i(U_i) = U$.

Remark 1.6.23. (i) Note that in the category of schemes arbitrary coproducts (colimits) are not guaranteed. However, we might consider the object $\bigsqcup_{i \in I} T_i$ in the category of sheaves. The above implies $\bigsqcup_{i \in I} T_i = T$.

(ii) Thus the first condition is can be seen as a generalisation of “faithfully flat”.

(iii) How would you define an fpqc-morphism $f: Y \to X$ of k-schemes?

Thus we have seen the following

Corollary 1.6.24. *A k-scheme X is a sheaf for the* fpqc *-topology.*

In other words: Every representable functor in Ens_k is a sheaf for the fpqc-topology.

2 Morphisms of k-schemes

2.1 Grothendieck's generic freeness lemma

In this section we study Grothendieck's generic freeness lemma.

Recall first the Noether normalisation lemma:

Lemma 2.1.1. *Let k be a field and R a k-algebra of finite type. Then there exists a subalgebra $A = k[x_1, \dots, x_d] \subset R$, which is a polynomial algebra such that R is a A-module of finite type.*

Remark 2.1.2. The integer d is uniquely determined and is the Krull dimension of R. If R is an integral domain, it is the transcendence degree of the fraction field of R.

In other words, when R is an integral domain, the length of every maximal chain of prime ideals of R is equal to the transcendence degree of its fraction field.

Recall that a topological space X is called catenary if for every pair of closed irreducible subsets $Y \subset Y'$ of X there is a maximal chain of closed irreducible subsets linking them, and any such chain has the same length.

Definition 2.1.3. Define catenary scheme and catenary ring.

Example 2.1.4. Give an example of a catenary scheme.

Example 2.1.5. Give an example of a non-catenary ring.

Lemma 2.1.6. *Let A be a Noetherian integral domain, B an A-algebra of finite type. Let M be a B-module of finite type. Then there is a nonzero element $a \in A$, such that $M[a^{-1}]$ is a free $A[a^{-1}]$-module.*

Proof. This can be proved by "dévissage" using Noether normalisation.
One possible outline:

- Recall: Let A be a commutetive ring with unit, and M an A-module, which is an increasing union of submodules M_n ($M_0 = 0$). Suppose that for all n, the quotient M_{n+1}/M_n is free. Then M is free.
- Show the statement for $A = B$.
- Reduce to the statement: is the statement is true for B, then it is true for the polynomial algebra $B[X]$ (Induction on the number of generators of B over A, Noether normalisation.)

- To show this, assume the statement is true for B and let M be a finitely generated $B[X]$-module. Let S be a finite generating set for M as $B[X]$-module. Let M_1 be the B-submodule of M generates by S. Define inductively B-submodule of M

$$M_{n+1} = M_n + XM_n.$$

  As a B-module, M is the increasing union of the M_n.

- For $n \gg 0$ the B-module M_n/M_{n-1} is isomorphic to M_{n+1}/M_n.
- There is $a \in A\backslash\{0\}$ such that $(M_{n+1}/M_n)[a^{-1}]$ is a free $A[a^{-1}]$-module for all n.
- $M[a^{-1}]$ is a free $A[a^{-1}]$-module.

□

There is also a constructive proof due to Blechschmidt in [2] which uses general topos theoretic techniques. It is based on the proposition that without loss of generality any reduced ring is a field.

Theorem 2.1.7. *Let A be a reduced commutative ring with unit, and B an A-algebra of finite type. If $f = 0$ is the only element of A such that*

(i) the $A[f^{-1}]$-modules $B[f^{-1}]$ and $M[f^{-1}]$ are free,

(ii) the $A[f^{-1}]$-algebra $B[f^{-1}]$ is of fintie presentation and

(iii) the $B[f^{-1}]$-module $M[f^{-1}]$ is finitely presented,

then $1 = 0$ in A.

We will only proof the finitely generated case.

Proposition 2.1.8. *Let A be a reduced ring and M a finitely generated A-module. If a is the only element of A, such that $M[a^{-1}]$ is a finite free $A[a^{-1}]$-module, then $A = 0$.*

Proof. First observe/prove the following:
Let M be an A-module with generating family $(x_1, \dots, x_n)$, assume that the only $g \in A$ such that one of the x_i is an $A[g^{-1}]$-linear combination of the others in $M[g^{-1}]$ is $g = 0$. Then M is free with basis $(x_1, \dots, x_n)$.

Main statement:
Induction on the length n of generating family of M. Formulate the induction hypothesis.
$n = 0$: (Assumption enters for $a = 1$).
$n \geqslant 1$: verify the assumptions of the observation. Here the induction hypothesis will be applied to and $A[g^{-1}]$-module $M[g^{-1}]$.
Follow that M is free over A. Apply the assumption to $a = 1$ again. □

The proof of the theorem is a combinatorial generalisation of this.
The usual version of Grothendieck's freeness lemmafollows from the above as a corollary.

2.2 Chevalley's constructability theorem

In this section, we consider the geometric version of schemes, i.e. a locally ringed space which admits a cover of affine spaces.

This section addresses the question what the topological image of a morphism of schemes looks like. As the following example illustrates, we have to restrict our attention to reasonable morphism of schemes in order to obtain a reasonable answer.

Example 2.2.1. Give an example of a morphism of schemes, where the image is ot a scheme.

However, even in a more restrictive case, the image of a scheme might not be a scheme.

Example 2.2.2. Consider $f : \mathbb{A}_2^{\mathbb{C}} \longrightarrow \mathbb{A}_2^{\mathbb{C}}$ given by $X \mapsto X, Y \mapsto XY$.

We first restrict our attention to morphisms of affine schemes with target irreducible and reduced, i.e. integral. The upcoming sequence of generalizations of the following proposition will lead eventually to the formulation and proof of Chevalleys theorem.

Proposition 2.2.3. *Let $f : Y \to X$ be a dominant morphism of finite type of affine noetherian schemes such that X is integral. Then the set of points $x \in |X|$ such that $f^{-1}(x) \neq \emptyset$ (the image) contains an open dense subset of X.*

Proof. Give a proof for this. □

Corollary 2.2.4. *Let $f : Y \to X$ be a dominant morphism of finite type between Noetherian schemes. Then $f(Y)$ contains an open dense subset of X.*

Proof. Give a proof for this. □

Definition 2.2.5. A subset of a topological space X is locally closed if it is the intersection of an open subset and a closed subset.

Remark 2.2.6. Intersecting the image of f with an irreducible component results in two cases: the generic point lies inside or outside the image. In both cases, the intersection is the union of a locally closed set with the intersection of $\mathrm{Im}(f)$ and a strictly smaller closed subset. This is due to the previous results. In fact, repeating those arguments, the image turns out to be a (finite) union of locally closed sets.

This observation leads to the following definition.

Definition 2.2.7. A constructible subset of a Noetherian topological space is a subset which belongs to the smallest family of subsets such that

(i) every open set is in the family,

(ii) a finite intersection of family members is in the family,

(iii) the complement of a family member is also in the family.

Lemma 2.2.8. *A subset of a Noetherian topological space is constructible if and only if it is the finite (disjoint) union of locally closed subsets.*

Proof. Prove this as a bonus. □

Images of schemes can be stranger than constructible.

Example 2.2.9. Show that any subset S of a scheme Y can be the image of a morphism. Or give a more concrete example of a morphism of schemes where the image is not constructible.

In all "reasonable" situations however the image of a scheme is a constructible set, which is made precise by Chevalley's theorem.

Proposition 2.2.10. *Let $f : Y \to X$ be a dominant morphism of finite type of affine integral schemes. Then the set of points $x \in |X|$ such that $f^{-1}(x) \neq \emptyset$ contains an open dense subset of X.*

Proof. Translate this into an algebraic statement. Apply Grothendieck's generic freenes. □

Corollary 2.2.11. *Let $f : Y \to X$ be a dominant morphism of finite type between Noetherian schemes. Then $f(X)$ contains an open dense subset of Y.*

Proof. Deduce this from the previous proposition □

Theorem 2.2.12. *Let X be a Noetherian scheme, $f : Y \to X$ a morphism of finite type. Then for every constructible subset $C \subset Y$, the image $f(C)$ is a construcible subset of X.*

Proof. Reduce to the case that $C = Y$ by showing:
Let Y be a Noetherian schem and C a constructible subset. Then there exists an affine scheme Y' and a morphism $f : Y' \to Y$ of finite type, such that $f(Y') = C$

Argue that it suffices to show that for every closed irreducible subset $Z \subset X$ whose generic point lies in $f(Y)$, $f(Y) \cap Z$ contains an open subset of Z by showing:
Let X be a Noetherian topological space. A subset $E \subset X$ is constructible if and only if for every closed irreducible subset $Z \subset X$, $E \cap Z$ contains a non-empty open subset of Z or is nowhere dense in Y.

Show this by applying Corollary 2.2.11 to $f^{-1}(Z) \to Z$. □

A variation of Chevalleys theorem regarding non-noetherian schemes relies on the fact that any ring is a filtered colimit of noetherian rings. This is made precise in the following remark.

Remark 2.2.13. Given any (commutative) ring R, for any finite set $S \subset R$, we get an induced ring extension $\mathbb{Z}[S]/I_S \subset R$. Thus, R is a filtered colimit of rings of this form, and, in particular, a filtered colimit of noetherian rings. Now, if $A = R[X_1, ..., X_n]/(f_1, ..., f_m)$ is an R-algebra of finite presentation, the coefficients of all f_i lie in some subring $\tilde{R} = \mathbb{Z}[S]/I \subset R$. Defining $\tilde{A} = \tilde{R}[X_1, ..., X_n]/(f_1, ..., f_m)$, we observe $A \cong \tilde{A} \otimes_{\tilde{R}} R$. In other words, there exists a cartesian square

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d] \\ \mathrm{Spec}(\tilde{A}) \arrow[r] & \mathrm{Spec}(\tilde{R}) \end{tikzcd}$$

with the bottom map of finite type and between noetherian schemes.

Theorem 2.2.14. *Let $f : Y \to X$ be a quasi-compact morphism of schemes which is locally of finite presentation. Then the image of every locally constructible set is locally constructible.*

Proof. Show this as a bonus. Can be deduced from the Noetherian case by a finite presentation trick. □

Remark 2.2.15. Chevalley's original formulation is a bit different [8, IV_1 Thm. 1.8.4]: Let $f : Y \to X$ be a finitely presented morphism of schemes. Then the image of any locally constructible subset of Y is a locally constructible subset of X.

2.3 Derivations

2.3.1 Kähler differentials

Definition 2.3.1. Let A be a k-algebra, M an A-module. A k-derivation $\partial_M : A \to M$ is a k-linear map satisfying the Leibniz formula

$$\partial_M(ab) = a\partial_M(b) + b\partial_M(a).$$

Denote by $\mathrm{Der}_k(A, M)$ the set of k-derivation from A to M.

In the following, we want to obtain the universal derivation $d : A \to \Omega^1_{A/k}$.

Definition 2.3.2. The module of relative Kähler differentials, denoted by $\Omega^1_{A/k}$ is an A-module together with a k-derivation $d : A \to \Omega^1_{A/k}$ which is universal with this property, that is for any k-derivarion $\partial_M : A \to M$ there is a unique A-module homomorphism $\Omega^1_{A/k} \to M$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "\partial_M"'] & \Omega^1_{A/k} \arrow[d] \\ & M \end{tikzcd}$$

commutes.

Construction 2.3.3. Define an A-module as a quotient of the free module $\bigoplus_{a\in A} A[a]$ such that it satisfies the desired universal property.

Kähler differentials are compatible with localisation.

Lemma 2.3.4. *Let A be a k-algebra.*

(i) For a multiplicative subset $S \subset k$ which maps to invertible elements of A, we have $\Omega^1_{A/k} \cong \Omega^1_{A/S^{-1}k}$.

(ii) For a multiplicative subset $S \subset A$ we have $S^{-1}\Omega^1_{A/k} \cong \Omega^1_{S^{-1}A/k}$

Proof. Give a short proof for this. □

As a consequence, it is possible to use the affine case to construct the sheaf of relative Kähler differentials globally (that is for schemes). However, there is another construction with a bit more geometric intuition that globalises immediately.

Proposition 2.3.5. *Let A be a k-algebra.*

(i) Let $\nabla : A \otimes_k A \to A$ be multiplication. The quotient $\ker\nabla/(\ker\nabla)^2$ has an A-module structure.

(ii) The map $\partial : A \to \ker\nabla/(\ker\nabla)^2$ given by $a \mapsto a\otimes 1 - 1\otimes a$ is a k-linear derivation.

(iii) There is a canonical isomorphism $\Omega^1_{A/k} \to \ker\nabla/(\ker\nabla)^2$ given by $\alpha \cdot d\beta \mapsto \alpha\otimes\beta - \alpha\beta\otimes 1$ compatible with the differentials.

Proof. Give a proof for this.
Hint for the last part: show that the morphism gives a map of k-derivations from A. By the universal property it mus be unique. Show that it ha an inverse given by $\alpha\otimes\beta \mapsto \alpha\cdot d\beta$.
Another possibility: Show that $\ker\nabla/(\ker\nabla)^2$ satisfies the universal property: for an A-module M there is an isomorphism

$$\begin{array}{rcl} \mathrm{Der}_k(A, M) & \cong & \mathrm{Hom}_A(\Omega^1_{A/k}, M) \\ \partial_M & \mapsto & (\alpha\otimes\beta \mapsto \alpha\cdot\partial_M\beta) \\ \phi\circ\partial & \leftarrow\!\shortmid & \phi \end{array}$$

□

Remark 2.3.6. Note that the map ∇ from above corresponds to the diagonal morphism $\Delta : \mathrm{Sp}(A) \to \mathrm{Sp}(A) \times \mathrm{Sp}(A)$.

More generally, suppose given a morphism of schemes $f : X \to S$. It induces the diagonal morphism

$$\Delta : X \to X \times_S X.$$

Then X is isomorphic with its image $\Delta(X)$ a locally closed subset of $X \times_S X$.

Definition 2.3.7. Give a global definition of Kähler differentials.

2.3.2 The total tangent space of an affine scheme

Let A be a k-algebra. In Example 1.5.9 we already considered the total tangent space as the following Weil restriction

$$T\mathrm{Sp}(A) = \prod_{k[\epsilon]/k} \mathrm{Sp}(A) \otimes_k k[\epsilon].$$

Remark 2.3.8. Describe this as a functor on k-algebras. For a k-algebra R, how do you give an R-point of $T\mathrm{Sp}(A)$?

Lemma 2.3.9. *There is a natural projection*

$$\pi : T\mathrm{Sp}(A) \to \mathrm{Sp}(A)$$

together with a zero section

$$0 : \mathrm{Sp}(A) \to T\mathrm{Sp}(A).$$

Proof. Determine both of these maps. □

We would like to describe $T\mathrm{Sp}(A)$ as an object of Ens_A, that is a functor $\mathrm{Alg}_A \to \mathrm{Ens}$. More precisely, for an A-algebra $\phi : A \to R$, we want to determine the set $T\mathrm{Sp}(A)(R, \phi)$.

Proposition 2.3.10. *The functor* $T\mathrm{Sp}(A)$ *is represented by an* A*-algebra.*

Proof. Think about what the set $T\mathrm{Sp}(A)(R, \phi)$ is by definition.
For an arbitrary element $\psi \in T\mathrm{Sp}(A)(R, \varphi)$ consider the difference with the element $\phi_0 : A \to R[\epsilon]$ that lies above ϕ via the zero section. Argue that for any $a \in A$, $\psi(a) - \phi_0(a)$ lies in the ideal (ϵ) of $R[\epsilon]$.
Use this to define a k-derivation $\partial\psi : A \to R$.

On the other hand, given a k-derivation $\partial : A \to R$, use ϕ_0 (lying above $\phi : A \to R$ via the zero section to obtain a map $\psi : A \to R[\epsilon]$.

Deduce

$$T\mathrm{Sp}(A)(R, \phi) \cong \mathrm{Der}_k(A, R).$$

Use the universal property of Kähler differentials, i.e. $\mathrm{Der}_k(A, R) \cong \mathrm{Hom}_A(\Omega^1_{A/k}, R)$, and Lemma 1.3.19 to identify the representing object. □

2.3.3 The total tangent space of schemes

Proposition 2.3.11. *Let X be a k-scheme of finite type. Then its total tangent space* $TX = \prod_{A/k}(X \otimes_k k[\varepsilon])$ *is representable by a k-scheme.*

Proof. Argue as in the proof of Proposition 1.5.12. □

Lemma 2.3.12. *Let $f : Y \to X$ be a morphism of k-schemes of finite type. The induced morphism on total tangent spaces $Tf : TY \to TX$ is compatible with the morphism of* $\mathscr{O}_X$*-modules* $\Omega^1_{X/k}|_Y \to \Omega^1_{Y/k}$.

Proof. Use the natural projections to obtain a commutative diagram

$$\begin{tikzcd} TY \arrow[r, "Tf"] \arrow[d, "\pi_Y"] & TX \arrow[d, "\pi_X"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \tag{2.3.1}$$

Obtain a morphism between the representing objects (as $\mathscr{O}_X$-algebras). Argue that in degree 1 this is exactly the morphism of $\mathscr{O}_X$-modules from the statement. □

2.3.4 The relative tangent space

In Defintion 1.5.3 we introduced Weil restriction for an affine extension of the base.

Definition 2.3.13. Extend this definition to an extension of arbitrary base schemes.

Definition 2.3.14. Let $fX \to S$ be a morphism of k-schemes of finite type. The relative tangent space of X/S is given by

$$T_{X/S} = \prod_{S[\epsilon]/S} X[\epsilon].$$

where $X[\epsilon] = X \otimes k[\epsilon]$ and similarly for S.

Lemma 2.3.15. *Let $\phi : \mathrm{Sp}(A) \to S$ be in Aff_S. Then $T_{X/S}(A, \phi)$ can be interpreted as the set of elements $\psi \in TX(A)$ which map to the 0-point of $TS(A)$.*

Proof. Argue that $T_{X/S}(A, \varphi)$ are those maps $\psi : \mathrm{Sp}(A[\epsilon]) \to X$, such that

$$\begin{tikzcd} \mathrm{Sp}(A[\epsilon]) \arrow[r, "\psi"] \arrow[d] & X \arrow[d] \\ \mathrm{Sp}(A) \arrow[r, "\phi"] & S \end{tikzcd}$$

commutes. □

Proposition 2.3.16. *There is a right exact sequence of $\mathscr{O}_X$-modules*

$$\Omega^1_{S/k}|_X \to \Omega^1_{X/k} \to \Omega^1_{X/S} \to 0.$$

Proof. Observe that for $B \in \mathrm{Aff}_X$, $T_{X/S}(B) \cong \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/S}, B)$.
From the previous lemma, obtain a cartesian diagram

$$\begin{tikzcd} T_{X/S} \arrow[r] \arrow[d] & X \arrow[d, "0"] \\ TX \arrow[r, "Tf"] & TS \times_S X \end{tikzcd}$$

Deduce an exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/S}, B) \to \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/k}, B) \to \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{S/k}, B)$$

Extend this to $\mathscr{O}_X$-modules and use Yoneda's Lemma. □

Example 2.3.17 (Euler sequence)**.** Let $X = \mathbb{P}^n_k$. There is a short exact sequence

$$0 \to \Omega^1_{X/k} \to \mathscr{O}_X(-1)^{n+1} \to \mathscr{O}_X \to 0.$$

Let R be a k-algebra. Recall that an R-point $x : \mathrm{Sp}(R) \to \mathbb{P}^n$ corresponds to an exact sequence (the tautological sequence)

$$0 \to N \to R^{n+1} \to L \to 0.$$

Describe the R-points $\tilde{x}$ in $T\mathbb{P}^n$ over the point x, that is $T\mathbb{P}^n(R, x)$ as exact sequences

$$0 \to \tilde{N} \to R[\epsilon]^{n+1} \to \tilde{L} \to 0$$

of (free) $R[\epsilon]$-modules that reduce to the above exact sequence. Argue that the data of $\tilde{N}$ is equivalent to the data of morphisms $\mathrm{Hom}_R(N, L) = N^* \otimes L$. Use that $T\mathbb{P}^n(R, x) = \mathrm{Hom}_R(\Omega^1_{\mathbb{P}^n} \otimes R, R)$ to identify

$$\Omega^1_{\mathbb{P}^n} \otimes R = N \otimes L^*$$

Thus $N = \Omega^1_{\mathbb{P}^n}(1) \otimes R$. From this deduce the Euler sequence.

2.4 Some properties of morphisms

2.4.1 Open morphisms

In the following going-up and going donw will play an important role, see Remark 1.6.8.

Definition 2.4.1. A morphism $f : Y \to X$ of k-schemes is called open, if the map on underlying topological spaces is open. It is universally open, if for any morphism of k-schemes $Z \to X$ the base change $f_Z : Y_Z \to Z$ is open.

Lemma 2.4.2. *A morphism $f : Y \to X$ of k-schemes which is flat and locally of finite presentation is open.*

Proof. Use Chevalley's constructibility theorem to show this.
This can be reduced to the affine case. One can even reduce to showing that the image of a standard affine is open. A flat ring map of finite presentation satisfies going down. By Chvevalleys theorem, the image of an open set is constructible (open sets are constructible). But this image is stable under generalisation because of going down. Then it follows that it is open by a classical topological argument. □

Corollary 2.4.3. *Flat morphisms of k-schemes which are locally of finite presentation are universally open.*

Proof. Note that the base change of flat morphisms is flat. □

2.4.2 Finite morphisms

Definition 2.4.4. A morphisms $f : Y \to X$ of k-schemes is finite if for every open affine $V = \mathrm{Sp}_k(A)$ of X, the preimage $U := f^{-1}(V)$ is open affine $U = \mathrm{Sp}_k(B)$ such that B is a finite A-algebra.

Lemma 2.4.5. *A finite morphism of k-schemes $f : Y \to X$ is proper.*

Proof. Verify the valuative criterion of properness for morphisms.
Reduce to the case that $X = \mathrm{Sp}_k(\mathscr{O}_K)$ where $\mathscr{O}_K$ is a dvr with fraction field K.
What does Y look like? What does a K-point of Y look like? How does it extend to an $\mathscr{O}_K$-point? □

2.4.3 Closed morphisms

Definition 2.4.6. A morphism $f : Y \to X$ of k-schemes is called closed, if the map on underlying topological spaces is closed. It is universally closed, if for any morphism of k-schemes $Z \to X$ the base change $f_Z : Y_Z \to Z$ is closed.

Lemma 2.4.7. *A proper morphism is universally closed.*

Proof. Give a short reason. Use the existence part of the valuative criterion of properness. □

Corollary 2.4.8. *Finite morphisms are universally closed.*

Remark 2.4.9. Note that a morphism of spectra $\mathrm{Spec}B \to \mathrm{Spec}A$ is closed if and only if the ring morphism $A \to B$ satisfies going up.

On the other hand, if $\mathrm{Spec}B \to \mathrm{Spec}A$ is open, then the ring morphism $A \to B$ satisfies going down, but the converse is only true if we assume in addition that $A \to B$ is of finite presentation.

2.4.4 Unramified, étale and smooth morphisms

Definition 2.4.10. Let $f : X \to S$ be a morphism of locally of finite presentation of k-schemes. Consider pairs (S', S'_0), where the S-scheme S' is the spectrum of a ring and $S'_0 \subset S'$ is a first order thickening. There is a restriction map

$$\mathrm{res}_{S'|S'_0} : \mathrm{Hom}_S(S', X) \to \mathrm{Hom}_S(S'_0, X).$$

The morphism f is said to be **unramified** / **smooth** / **étale** if $\mathrm{res}_{S'|S'_0}$ is injective / surjective / bijective.

Remark 2.4.11. Restate this in terms of commutative diagrams

Remark 2.4.12. To check the conditions unramified / smooth / étale it suffices to consider test diagrams of certain types:

(i) It suffices to assume that $S'_0 \to S'$ is induced by a morphism of **local rings** such that the kernel is a square zero ideal (the conditions are local on the source).

(ii) It suffices to assume that $S'_0 \to S'$ is induced by a morphism of **Artinian local rings** such that the kernel is of length 1.

(iii) It suffices to assume that $S'_0 \to S'$ is induced by a morphism of **Artinian local rings** such that the kernel is of length 1 and $S' \to S$ **is of finite type**.

(iv) It suffices to assume that $S'_0 \to S'$ is induced by a morphism of **Artinian rings** such that the kernel is a square zero ideal.

Remark 2.4.13. Let $f : X \to S$ be a morphism of finite type of k-schemes. The following are equivalent:

(i) The morphism f is étale.

(ii) The morphism f is smooth and unramified.

(iii) The morphism f is flat and unramified.

The first of the equivalences is rather straight forward from the definition, whil the second requires more work.

For the rest of this section, let k be a field.

Proposition 2.4.14. *Let $f : X \to S$ be a morphism of finite type of k-schemes. Let k'/k be an Artin ring. Let $s_0 : \mathrm{Sp}(k') \to S$ be a k'-point of S and $s : \mathrm{Sp}(k'[\epsilon]) \to S$ a tangent vector at s_0. Let x_0 be a k'-point of X over s_0.*

(i) If f is étale the morphism $T_{x_0}X(k') \to T_{s_0}S(k')$ is an isomorphism.

(ii) If f is smooth, the morphism $T_{x_0}X(k)' \to T_{s_0}S(k')$ is surjective.

(iii) The morphism f is unramified if and only if the morphism $T_{x_0}X(k') \to T_{s_0}S(k')$ is injective for all k'-points.

Proof. Show this.
Consider the diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & \mathrm{Sp}(k') \arrow[l, "x_0"] \arrow[d] \\
S & \mathrm{Sp}(k'[\epsilon]) \arrow[l, "s"'] \arrow[ul, dashed, "x"]
\end{tikzcd}$$

For the last part, one can argue that this implies $\Omega_{X/S} = 0$. Then for two dottes morhisms, one has to show that they coincide. Look at the induced morphism to $S' \to X \times_S X$. Argue that this map restricted to S'_0 factors through the diagona Δl. COnclude using the observation that $\ker \Delta = (\ker \Delta)^2$ and the $S'_0 \to S'$ is defined by a square zero ideal. $\square$

Example 2.4.15. Find an example that shows that for the first and second item of the proposition, the converse is not true.
Assume $\mathrm{char}(k) \neq 2$. Consider the projection $f : k[x] \to k[x]/(x^2)$. Argue that this induces an isomorphism on tangent spaces:
For a k-point in each case is a morphism

$$a : k[x] \to k \qquad\qquad b : k[x]/(x^2) \to k$$

and uniquely determined by the image of x. Consider tangent vectors over such points

$$\begin{tikzcd} k[x] \arrow[r] \arrow[dr] & k[\epsilon] \arrow[d] \\ & k \end{tikzcd} \qquad \begin{tikzcd} k[x]/(x^2) \arrow[r] \arrow[dr] & k[\epsilon] \arrow[d] \\ & k \end{tikzcd}$$

Note that $k[x] \to k[\epsilon]$ factors through $k[x]/(x^2)$. Deduce that

$$Tf : T_b\mathrm{Sp}(k[x]/x^2)(k) \to T_a\mathrm{Sp}(k[x])(k)$$

is an isomorphism.
Show that f is not flat.

However we have the following:

Corollary 2.4.16. *A morphism of finite type $f : X \to S$ of k-schemes is étale if and only if is is flat the induced morphism on the total tangent space*

$$Tf : TX \to X \times_S TS$$

is an isomorphism.

Proof. Give a proof for this. □

Lemma 2.4.17. *A morphism of finite type $f : X \to S$ of k-schemes is smooth if for every $x \in X$ there is an open neighbourhood $x \in U \subset X$ and an étale morphism*

$$g : U \to S[s_1, \dots, s_n] := S \otimes_k k[s_1, \dots, s_n]$$

such that $f|_U = \pi \circ g$ where $\pi : S[s_1, \dots, s_n] \to S$ is the canonical morphism.

Proof. Use that π is smooth. □

Remark 2.4.18. This was Grothendieck's first definition of smoothness. Showing the equivalence requires some work.

Lemma 2.4.19. *A morphism of finite type $f : X \to S$ of k-schemes which is smooth or étale is open.*

Proof. Note that f is in particular flat and hence open. □

We will now consider smooth schemes over the field k.

Lemma 2.4.20. *Let $X = \mathrm{Sp}(A)$ where A is a k-algebra of finite type. Then X/k is smooth if and only if $\Omega^1_{A/k}$ is a locally free A-module.*

Proof. Show that $\Omega^1_{A/k}$ is projective and finite:
Choose a surjective morphism $P = k[x_1, \ldots, x_n] \to A$ with kernel J. Show that the surjeciton $P/J^2 \to A$ has a section $A \to P/J^2$. Deduce that there is a split exact sequence

$$0 \to J/J^2 \to \Omega^1_{P/k} \otimes_P A \to \Omega^1_{A/k} \to 0$$

The converse involves more computations. If $\Omega^1_{A/k}$ is projective, then the above sequence splits. Then a longer calculation is needed to obtain a section of $P/J^2 \to A$. □

Remark 2.4.21. For morphisms between smooth schemes, we have a converse of the frist two points in Proposition 2.4.14:
Let $f : X \to S$ be a morphism of finite type of smooth k-schemes.

(i) The morphism f is étale if and only if $Tf : TX \to X \times_S TS$ is an isomorphism.

(ii) The morphism f is smooth if and only if Tf is surjective.

Let $X = \mathrm{Sp}(A)$ where A is a k-algebra of finite type. Let $f_1, \ldots, f_n \in A$ and

$$f : B = k[x_1, \ldots, x_n] \to A$$

be the unique k-algebra morphism such that $f(x_i) = f_i$. This induces a morphism of A-modules

$$df : \Omega^1_{B/k} \otimes_B A \to \Omega^1_{A/k}$$

with $dx_i \mapsto df_i$.

Lemma 2.4.22. *The morphism f above is étale if and only if df is an isomorphism and the local rings of A at maximal ideals have dimension n.*

Proof. Show this as a bonus.
Observe that the first condition means $\Omega_{A/k[x_1,\ldots,x_n]} = 0$, which is equivalent to f being unramified. Use a relation between flatness and local dimension(for regular local rings). □

2.5 Dimension

Definition 2.5.1. Let X be a k-scheme. The dimension of $|X|$ is the maximal length of chains

$$Y_1 \subset Y_2 \subset \cdots \subset Y_d$$

where $Y_1, \ldots, Y_d$ are irreducible closed subsets of $|X|$.

Remark 2.5.2. For an affine scheme $X = \mathrm{Sp}_k(A)$ the chains of irreducible closed subsets of $\mathrm{Spec}(A)$ correspond bijectively to chains of prime ideals of A. Hence its Krull dimension is equal to the dimension of its underlying topological space.

Definition 2.5.3. Let X be a k-scheme. For any $x \in |X|$, denote by $\dim_x(X)$ the dimension of the local ring $\mathscr{O}_{X,x}$. This is often called the local dimension of X at x.

Remark 2.5.4. (i) Let X be locally Noetherian. The number $d = \dim(\mathscr{O}_{X,x})$ is the smallest integer such that there are $f_1, \ldots, f_d \in \mathfrak{m}_x$ such that $(f_1, \ldots, f_d)$ contains a power of $\mathfrak{m}_x$.

(ii) The assignment

$$|X| \to \mathbb{N}, x \mapsto \dim_x(X)$$

defines an upper semi-continuous function.

(iii) We have $\dim(X) = \max_x(\dim_x(X))$.

We describe now the dimension of fibres along morphisms.

Lemma 2.5.5. *Let $f : Y \to X$ be a morphism of k-schemes, $y \in |Y|$ and $x = f(y) \in |X|$. Then we have an inequality*

$$\dim_y(Y) \leqslant \dim_y(Y_x) + \dim_x(X)$$

and equality if $\mathscr{O}_{Y,y}$ is a flat $\mathscr{O}_{X,x}$-algebra.

Proof. For the inequality use the characterisation of local dimension from the remark above. For the equality in the flat case, use the 'going down' for open morphisms. $\square$

3 Basic structure of group schemes

3.1 Definitions

Let k be a (commutative) ring (with unit).

Definition 3.1.1. A k-group scheme (or an algebraic group over k) is an element in Gr_k, that is a functor

$$G : \mathrm{Alg}_k \to \mathrm{Gr}$$

which as a functor to Ens, that is as an element in Ens_k, is (represented by) a k-scheme.

A morphism of group schemes is a morphism in Gr_k.

Remark 3.1.2. This is equivalent to saying that G is a k-scheme with morphisms

- a morphism $e : \mathrm{Sp}(k) \to G$ called unit
- a morphism $\iota : G \to G$ called inversion
- a morphism $\mu : G \times G \to G$ called composition law (or multiplication)

such that for any k-algebra A these data endow $G(A)$ with an abstract group structure.

This is exemplified by the following commutative diagrams:

Provide the commutative diagrams that describe the group scheme structure.

We often describe the morphisms e, ι, μ on A-points.

Remark 3.1.3. Use the second definition to define when group scheme is trivial and when a homomorphism is trivial.

Remark 3.1.4. Similarly one can define a k-monoid scheme (or an algebraic monoid over k).

Give a useful definition of k-monoid scheme.

Example 3.1.5. Describe the group schemes $\mathbb{G}_m$ and $\mathbb{G}_a$ to ilustrate the definition.

Definition 3.1.6. Let G be a k-group scheme and X a k-scheme. An action of G on X is a morphism

$$\sigma : G \times X \to X$$

which defines for each k-algebra A an action of the abstract group $G(A)$ on the set $X(A)$.

Remark 3.1.7. This means that the morphism σ fits in the following commutative diagrams:

Provide the commutative diagrams that describe the group action.

Example 3.1.8. There is an action of $\mathbb{G}_m$ on $\mathbb{G}_a$ as follows:
Describe this action

Remark 3.1.9. Let $\rho : H \to G$ be a morphism of k-group schemes, X a k-scheme with an action by G. Then there is an action of H on X as follows:
Describe the action of H on X.

Definition 3.1.10. A k-group scheme G can act on itself in three different ways:

- from the left, given for a k-algebra A by
$$G(A) \times G(A) \to G(A), (g, h) \mapsto gh,$$
- from the right, given for a k-algebra A by
$$G(A) \times G(A) \to G(A), (g, h) \mapsto hg^{-1},$$
- by conjugation , given for a k-algebra A by
$$G(A) \times G(A) \to G(A), (g, h) \mapsto ghg^{-1}.$$
The conjugation preserves the grop structure of G.

3.2 The Lie algebra of a group scheme

Recall the definition of a Lie algebra over a field:

Definition 3.2.1. A Lie algebra over a field F is a F-vector space $\mathfrak{g}$ with a binary operation
$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
called Lie bracket which satisfies

(i) bilinearity
$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y]$$
for all $a, b \in F$ and $x, y, z \in \mathfrak{g}$,

(ii) alternativity
$$[x, x] = 0$$
for all $x \in \mathfrak{g}$,

(iii) the Jacobi identity
$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$
for all $x, y, z \in \mathfrak{g}$.

Remark 3.2.2. One can extend this definition to commutative rings with unit (and even non-commutative rings). However some of the classical results such as the Poincaré–Birkhoff–Witt theorem might not hold anymore.

We will encounter a generalisation of this in our context.

3.2.1 The Lie algebra functor

Definition 3.2.3. Let G be a k-group scheme. The Lie algebra of G is defined as the product

$$\mathrm{Lie}(G) = TG \times_{\pi,G,e} \mathrm{Sp}(k),$$

where $\pi : TG \to G$ is the canonical projection of the tangent space, and $e : \mathrm{Sp}(k) \to G$ is the uni section of G.

Lemma 3.2.4. *The functor* $\mathrm{Lie}(G)$ *is represented by a* k*-algebra.*

Proof. Deduce from the definition that there is an isomorphism

$$\mathrm{Lie}(G) \cong \mathrm{Sp}(\mathrm{Sym}_k(e^*\Omega^1_{G/k})).$$

. ☐

Corollary 3.2.5. *If* k *is a field, then* $\mathrm{Lie}(G)$ *takes values in* k*-vector spaces.*

Lemma 3.2.6. *More generally,* $\mathrm{Lie}(G)$ *is a group scheme.*

Proof. We could have defined $\mathrm{Lie}(G)$ as the functor

$$\mathrm{Alg}_k \to \mathrm{Ens}, A \mapsto \mathrm{Hom}_k(e^*\Omega^1_{G/k}, A).$$

☐

Remark 3.2.7. We can define the Lie algebra of a general k-group functor, that is an element in Gr_k which is not necessarily representable. However, then this Lie algebra is also in general not representable.

Lemma 3.2.8. *There is an isomorphism in* Ens_k

$$TG \cong G \times_k \mathrm{Lie}(G).$$

Proof. Note that for any k-algebra A, there is a short exact sequence (of groups)

$$1 \to \mathrm{Lie}(G)(A) \to G(A[\epsilon]) \to G(A) \to 1$$

☐

Lemma 3.2.9. *Let* $\alpha : H \to G$ *be a morphism of* k*-group schemes. It induces a morphism* $d\alpha : \mathrm{Lie}(H) \to \mathrm{Lie}(G)$*.*

Proof. Give a short explanation. ☐

Consequently, we can interprete Lie as a functor from k-group schemes to k-module schemes.

Example 3.2.10. Let M be a k-module of finite type. Set $\mathbb{V}_k(M) = \mathrm{Sp}(\mathrm{Sym}_k(M))$. Recall that $\mathbb{V}_k(M)$ is a k-group scheme and determine $\mathrm{Lie}(\mathbb{V}_k(M))$. We call a group scheme good, if it is isomorphic to its Lie algebra. This example shows that for a k-group scheme G, $\mathrm{Lie}(G)$ is good.

Example 3.2.11. Let M be a k-module of finite type. Consider the functor

$$\mathrm{GL}\,(M) : \mathrm{Alg}_k \to \mathrm{Ens}\,, A \mapsto \mathrm{Aut}_A(M \otimes_k A).$$

Argue that this is a k-group scheme if M is a locally free k-module.
Show that its Lie algebra is the functor

$$\mathfrak{gl}(M) : \mathrm{Alg}_k \to \mathrm{Ens}\,, A \mapsto \mathrm{End}_A(M \otimes_k A).$$

3.2.2 The adjoint representation of a group scheme

We will now describe a natural action of G on $\mathrm{Lie}(G)$.

Lemma 3.2.12. *Let A be a k-algebra. The group $\mathrm{Lie}(G)(A)$ is a normal subgroup of $G(A[\epsilon])$ and $G(A[\epsilon])$ acts on $\mathrm{Lie}(G)(A)$ by conjugation.*

Proof. Explain this briefly. □

Corollary 3.2.13. *This induces an action of G on $\mathrm{Lie}(G)$ called the adjoint action. In other words,for every k-algebra A a map*

$$\rho : G(A) \to \mathrm{Aut}(\mathrm{Lie}(G)(A)).$$

Proof. Note that for a k-algebra A, the inner action of $\mathrm{Lie}(G)(A)$ on itself is trivial. It induces an action of $G(A)$ on $\mathrm{Lie}(G)(A)$ which is natural in A. □

Lemma 3.2.14. *For any k-algebra A, the map ρ maps into $\mathrm{Aut}_A(\mathrm{Lie}(G)(A))$.*

Proof. Show that the following formulas hold for $g \in G(A)$, $a \in A$, $x, x' \in \mathrm{Lie}(G)(A)$:

$$\rho(g)(x + x') = \rho(g)(x) + \rho(g)(x')$$
$$\rho(g)(cx) = c(\rho(g)(x))$$

For the second one, use that there is an action of A on $\mathrm{Lie}(G)(A)$:
$c \in A$ defines an A-algebra morphism

$$u_c : A[\epsilon] \to A[\epsilon], a + \epsilon b \mapsto a + \epsilon cb$$

which is the identity modulo ϵ and he nce induces a commutative diagram

$$\begin{tikzcd} G(A[\epsilon]) \arrow[r, "G(u_c)"] \arrow[d, "\pi"] & G(A[\epsilon]) \arrow[d, "\pi"] \\ G(A) \arrow[r, "\mathrm{id}"] & G(A) \end{tikzcd}$$

□

Remark 3.2.15. The proof of this lemma shows that for a k-algebra A the group $\mathrm{Lie}(G)(A)$ is in fact an A-module. In particlar, we can regard $\mathrm{Lie}(G)$ as a $\mathscr{O}_k$-module.

Example 3.2.16. For a k-algebra A, describe the action of A on $\mathfrak{gl}_n = \mathrm{Lie}(\mathrm{GL}_n)$.

Lemma 3.2.17. *Let $f : H \to G$ be a morphism of k-group schemes, and $df : \mathrm{Lie}(H) \to \mathrm{Lie}(G)$ the induced morphism. The adjoint action ρ is compatible with f and df.*

Proof. Show that for a k-algebra A there is a commutative diagram

$$\begin{tikzcd} H(A) \times \mathrm{Lie}(H)(A) \arrow[r] \arrow[d, "f \times df"] & \mathrm{Lie}(H)(A) \arrow[d, "df"] \\ G(A) \times \mathrm{Lie}(G)(A) \arrow[r] & \mathrm{Lie}(G)(A) \end{tikzcd}$$

□

We will now use the adjoint map ρ to define a Lie bracket on $\mathrm{Lie}(G)$. For this we first have the following observation.

Lemma 3.2.18. *The adjoint action of G on $\mathrm{Lie}(G)$ defines a homomorphism of k-group schemes*

$$\rho : G \to \mathrm{GL}\,(e^*\Omega^1_{G/k}).$$

Proof. Use the formula $\mathrm{Lie}(G)(A) \cong \mathrm{Hom}_k(e^*\Omega^1_{G/k}, A)$. □

Applying the functor Lie to this, we obtain the following

Corollary 3.2.19. *There is a morphism of k-group schemes*

$$d\rho : \mathrm{Lie}(G) \to \mathfrak{gl}(e^*\Omega^1_{G/k}).$$

As a consequence, we obtain for every k-algebra A, a morphism

$$d\rho : \mathrm{Lie}(G)(A) \to \mathrm{End}_A(\mathrm{Lie}(G)(A)).$$

Definition 3.2.20. We call Lie bracket of $\mathrm{Lie}(G)$ the operation $[-,-] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ which is given for any k-algebra A by

$$[-,-] : \mathrm{Lie}(G)(A) \times \mathrm{Lie}(G)(A) \to \mathrm{Lie}(G)(A), (x, y) \mapsto d\rho(x)(y).$$

Example 3.2.21. Obtain the Lie bracket on $\mathfrak{gl}_n = \mathrm{Lie}(\mathrm{GL}_n)$ from the definition. Thus, if k is a field and F/k a field extension, the F-algebra $\mathfrak{gl}_n(F)$ is a Lie algebra in the classical sense.

Example 3.2.22. More generally, for a locally free k-module M of finite type, the Lie bracket on $\mathfrak{gl}(M)$ is given by the commutator.

Remark 3.2.23. It turns out, that for a good k-group scheme H, a similar statement is true: in this case we can identify $\mathrm{Lie}(\mathrm{Aut}_{\mathscr{O}_k}(H)) \cong \mathrm{End}_{\mathscr{O}_k}(H)$, that is for a k-algebra A, $\mathrm{Lie}(\mathrm{Aut}_{\mathscr{O}_k}(H))(A) \cong \mathrm{End}_A(H(A))$. Then by a similar computation as in the example, one can show that the bracket corresponds on th right hand side to the commutator

$$[X, Y] = X \circ Y - Y \circ X.$$

Proposition 3.2.24. *In the situation above there is a relation*

$$d\rho([x, y]) = d\rho(x) \circ d\rho(y) - d\rho(y) \circ d\rho(x).$$

Proof. Use that $\mathrm{Lie}(G)$ is a good k-group scheme, and that therefore the bracket on $\mathrm{End}_A(\mathrm{Lie}(G)(A))$ is given by the commutator by the above remark and Example 3.2.22. □

Corollary 3.2.25. *The bracket* $[-,-]$ *on* $\mathrm{Lie}(G)$ *satisfies the Jacobi identity.*

Proof. Apply $d\rho([x, y])$ from the proposition to another element z. □

Remark 3.2.26. As a consequence we see that for a field extension F/k, $\mathrm{Lie}(G)(F)$ is a Lie algebra in the classical sense. In particular, for a field k we obtained a functor $\mathrm{Lie}(-)(k)$ from k-group schemes to Lie algebras over k.

3.3 The identity component

Recall that a topological space X is called connected, if its only open-closed subsets are the whole space X and the empty set $\emptyset$. The maximal connected subsets of a topological space are called connected components. We denote by $\pi_0(X)$ the set of connected components. Every topological space X has a decomposition

$$X = \bigsqcup_{\alpha \in \pi_0(X)} X_\alpha$$

into connected components indexed by the set $\pi_0(X)$.

A continuous map $f : X \to Y$ of topological spaces induces a map $\pi_0(X) \to \pi_0(Y)$, because for every $\alpha \in \pi_0(X)$, there is a unique element $\beta \in \pi_0(Y)$ such that $X_\alpha \subset f^{-1}(Y_\beta)$.

3.3.1 The identity component of a group scheme over a field

In this section let k be a field.

Definition 3.3.1. For a k-group scheme G, we denote by G° the connected component containing the neutral element $e \in G(k)$ and call it the identity component.

Proposition 3.3.2. *The identity component* G° *of a* k*-group scheme* G *which is locally of finite type is a* k*-subgroup scheme of* G *which is geometrically connected.*

To show this, we will need the following statement:

Lemma 3.3.3. *Let X and Y be two connected k-schemes (of finite type). Assume that $X(k)$ is non-empty. Then $X \times_k Y$ is connected.*

Proof. Assume first that we are in the affine case $X = \mathrm{Sp}_k(A)$, $Y = \mathrm{Sp}_k(B)$.
Assume the $X \times_k Y = U \sqcup V$ is not connected. In particular the representing k-algebra of $X \times_k Y$ can be written as a product $A \otimes_k B = C \times D$.
Consider the projection $U \to X$ and argue that the image of $|U|$ is open in $|X|$ using that $X \times Y \to X$ is flat of finite type.
Next show that the image of U is closed. For this note that it is constructible and show that it is closed under specialisation. Observe, that a point $x \in |X|$ is in the image of U if $C \otimes \kappa(x) \neq 0$. To show that $y \in \overline{\{x\}}$ is in the image of U, consider the localisation $C \otimes A_y$. Use Kaplansky's theorem to show that it is free and deduce that $C \otimes \kappa(y) \neq 0$.
This shows that $|U| \to |X|$ is surjective.
Let now $x \in X(k)$. Argue that the intersection of U and $\{x\} \times Y$ is a non-empty open-closed subset of $\{x\} \times Y$ and hence equal to it. Thus $\{x\} \times Y \subset U$.
Similarly for V. Derive a contradiction.

To reduce to the affine case: we first reduce to the case that Y is affine. Assume we know that $X \times_k Y$ is connected in the case when Y is affine (and X not necessarily). If Y is not affine, for each affine $P \subset Y$, we know that $X \times P$ is connected. Let $x \in X(k)$, then $\{x\} \times Y$ is connected and the intersection with each $X \times P$ is not empty. Hence $\{x\} \times Y$ intersects all the connected components of $X \times Y$ and $X \times Y$ must be connected.
To reduce to the case that X is also affine, note that the projection $X \times \mathrm{Spec} B \to X$ is affine, that is, the preimage of ever affine is again affine. But the arguments that we used above to show that U is open and closed are local on the target.

□

Example 3.3.4. Consider the $\mathbb{R}$-schemes $X = Y = \mathrm{Sp}_{\mathbb{R}}(\mathbb{C})$. Show that $X \times_{\mathbb{R}} Y$ is not connected. Conclude that the condition of the existence of a k-point in the lemma above is necessary.

We come now to the proof of the proposition.

Proof. Note that $G^\circ(k)$ is by definition not empty. Hence $G^\circ \otimes_k k'$ is connected for every finite extension k'/k. Deduce that G° is geometrically connected.

To show that G° is a k-group scheme, show that multiplication and inverse restricts from G to G°. Use that they are morphisms of schemes, thus continuous and map connected components to connected components. □

Corollary 3.3.5. *A k-group scheme G is connected if and only it is geometrically connected.*

Proof. Note that G is connected if and only if $G = G^\circ$.
Use that for a finite extension k'/k one has $(G^\circ)_{k'} \cong (G_{k'})^\circ$. □

Remark 3.3.6. We will see that in the situation of Proposition 3.3.2 that G° is (geometrically) irreducible.

Moreover, in this case, it is quasi-compact and hence of finite type.

Example 3.3.7. Give examples of connected group schemes.

Example 3.3.8. Give an example of a non-connected group - think for example of direct products of groups.

Example 3.3.9. As a more interesting example we want to see that the orthogonal group is not connected. Let k be a field of characteristic $\neq 2$ and V a k-vector space of dimension n with a symmetric bilinear form $B : V \times V \to k$. The orthogonal group $\mathrm{O}\,(V, B)$ is the functor

$$\mathrm{Alg}_k \to \mathrm{Gr}\,, A \mapsto \{g \in \mathrm{GL}\,(V \otimes_k A) \mid B(gx, gy) = B(x, y)\}.$$

Argue that this is a closed subgroup of the $\mathrm{GL}\,(V)$.
Observe that the determinant $\bigwedge^n V$ of V is also equipped with a symmetric bilinear form $\bigwedge^n B$. The determinant defines a surjective homomorphism

$$\det : \mathrm{O}\,(V, B) \to \mathrm{O}\,(\bigwedge^n V, \bigwedge^n B).$$

Show that the group $\mathrm{O}\,(\bigwedge^n V, \bigwedge^n B)$ is the subgroup $\mu_2 = \{\pm 1\} \subset \mathbb{G}_m$ and deduce that $\mathrm{O}\,(V, B)$ has two connected components.
The identity component of $\mathrm{O}\,(V, B)$ is the special orhtongonal group

$$\mathrm{SO}\,(V, B) = \{g \in \mathrm{O}\,(V, B) \mid \det(g) = 1\}.$$

3.3.2 The identity component of a smooth group scheme

Let S be a k-scheme and G a smooth S-group scheme of finite type. For a point $s \in |S|$ consider the identity component G_s° of the fiber of G over s.

Proposition 3.3.10. *The union $\bigcup_{s \in |S|} |G_s^\circ|$ is Zariski open in G. The corresponding open subscheme, denoted by G_S° is a smooth S-group scheme whose fibres are connected algebraic groups.*

Proof. A more general statement can be found in [8, IV.3 Prop. 15.6.4]. Give a sketch of the proof.
Note that $E := \bigcup_{s \in |S|} |G_s^\circ|$ is constructible:. Let $x \in G$ be a point and $Z \subset G$ the reduced subscheme with underlying space $\overline{\{x\}}$, similarly let $Y \subset S$ be the reduced subscheme with underlying space $\overline{f(Z)}$, where f is the structure morphism.
As $f|_Z$ factors through Y, we may replace S by Y, and assume $f(x) = \eta =$ the generic point of the integral scheme S.
By hypothesis, G_η is union of two open subschemes G_η° and G_η^1, induced on open complementary subsets. If we replace S if necessary by an open neighbourhood of η, we

may assume that G is union of two disjoint opens G^0 and G^1, such that $G^i_\eta = (G^i)_\eta$. Es $e : S \to G$ is continues and injective, S is the union of two open disjoints $e^{-1}(G^i)$. But S is irreducible and $e(\eta) \in X^0_\eta$, we have that $e^{-1}(X^1) = \emptyset$. Thus $e : S \to G^0 = G^\circ_S$. As $(G^0)_\eta$ is geometricall connected, the same is true for any $s \in \overline{\{\eta\}}$. As $e(s) \in (G^0)_s$ we have $(G^0)_s = G^0_s$

Thus it suffices to show that it is stable under generalisation.

It is possible to reduce to the case that S is the spectrum of a discrete valuation ring: let $g, g' \in |G|$ with $g \in \bigcup_{s\in|S|} |G^\circ_s|$ and $itg \in \overline{\{g'\}}$. Consider the images s, s' of g and g' in S. There is a morphism from the spectrum of a discrete valuation ring into S such that the image of the closed point is s, and the image of the generci point is s'.

Let η_s be the generic point of the special fibre G_s. As $G \to S$ is smooth $\mathscr{O}_{G,\eta_s}$ is also a dvr, in particular reduced. One can show that in this case, $G_{s'}$ is connected.

It follows that $\bigcup_{s\in|S|} |G^\circ_s| = G^\circ$ which is by definition the connected component of G containing S. □

3.3.3 The set of connected components

Let k be a field, $\overline{k}$ an algebraic closure, and G a k-group scheme of finite type.

Lemma 3.3.11. *The set $\pi_0(G \otimes_k \overline{k})$ is a finite group.*

Proof. Note that $\pi_0(G \otimes_k \overline{k})$ can be seen as a quotient of $G_{\overline{k}}$ and $G^\circ_{\overline{k}}$. □

Remark 3.3.12. There is an action of $\mathrm{Gal}\,(\overline{k}/k)$ on $\pi_0(G \otimes_k \overline{k})$. The set $|\pi_0(G)|$ can be seen as the set of orbites of $\mathrm{Gal}\,(\overline{k}/k)$ on $\pi_0(G \otimes_k \overline{k})$.

Let S be a scheme over k and G an S-group scheme. For $s \in |S|$, the fibre G_s is an algebraic group over the residue field $\kappa(s)$. Consider the group of connected components $\pi_0(G_s)$ which can vary with s.

Example 3.3.13. Let $Y \to X$ be a a covering of degree 2 which is flat and ramified. There is a norm

$$\prod_{Y/X} \mathbb{G}_{m,Y} \to \mathbb{G}_{m,X}.$$

Describe this map.
Let T be the kernel. Explain that over the ramification locus, the fibres of T have two connected components

Remark 3.3.14. $\pi_0(G)$ is represented by a non-separated scheme.
If you are interested, look more into this to explain it.

3.4 The translation argument

In this section, let k be a field.

Lemma 3.4.1. *Let G be a k-group scheme and U, V dense opens in G. Then $UV = G$.*

Proof. As multiplication is surjetive, we have to show that it is still surjective when restriced to $U \times_k V$. Argue that it sufficed to show that for every algebraic closure $\overline{k}$ the map $U(\overline{k}) \times V(\overline{k}) \to G(\overline{k})$ is surjective. Use the translation argument to show this. □

Corollary 3.4.2. *Let $\rho : G \to H$ be a morphism of connected algebraic groups over k. The image of ρ is a closed.*

Proof. We know that the image is a subgroup scheme of H. Its closure is a closed sub group scheme. Argue with Chevalley's theorem that the image of ρ contains a dense open of its closure. Then use the lemma above. □

Corollary 3.4.3. *Let H be a k-subgroup scheme of G, then H is closed.*

Another consequence of the lemma of two dense opens is the following:

Corollary 3.4.4. *Let G be an irreducible k-group scheme. Then G is quasi-compact.*

Proof. Note that any open (non-empty) affine of G is dense. Apply Lemma 3.4.1 with $U = V$. □

Lemma 3.4.5. *Let G be an algebraic connected group over k (that is, a group scheme which is of finite type). Then G is irreducible.*

Proof. Observe that it suffices to show that G is geometrically irreducible. Thus let k be algebrically closed. Show that in this case, the automorphism group of G acts transitively on $|G|$. Use this to show that the irreducible components are disjoint. As G is connected, there can be only one irreducible component. □

Lemma 3.4.6. *Let G be an algebraic connected group over k (that is, a group scheme which is of finite type). G is reduced as a k-scheme if and only if G is smooth.*

Proof. Note that any reduced algebraic scheme is geometrically reduced. Argue that it suffices to treat the case that k is algebraically closed. There is a smooth open $U \subset G$. By a similar argument as above G can be covered by the translates of U. Conclude that G is smooth. □

Corollary 3.4.7. *Let G be a k-group scheme which is locally of finite type. Then G° is (geometrically) irreducible, quasi-compact and hence of finite type.*

Corollary 3.4.8. *Let G be a k-group scheme which is locally of finite type. Every connected component of G is irreducible, of finite type and has same dimension as G°.*

4 Actions of group schemes

4.1 The fixed subscheme

Let G be a (smooth) k-group scheme and X a separated k-scheme. Let $G \times X \to X$ be an action by G on X. Consider the functor of fixed points

$$X^G : R \mapsto X^G(R) = \{x \in X(R) \mid gx_{R'} = x_{R'} \forall g \in G(R'), R' \in \mathrm{Alg}_R\}.$$

We want to see that X^G is represented by a k-scheme, more precisely by a closed subscheme of X. We wil deduce this from a more general statement.

Theorem 4.1.1. *Let G and X be as above. The functor X^G is representable by a closed subscheme of X.*

Proof. Explain how a point $x \in X(R)$ defines two maps $G(R) \to X(R)$, and hence tow maps

$$X \rightrightarrows \mathrm{Hom}(G, X)$$

Together this gives a commutative diagram

$$\begin{tikzcd}
X \arrow[r] & \mathrm{Hom}(G,X) \times \mathrm{Hom}(G,X) \arrow[r, "\sim"] & \mathrm{Hom}(G, X \times X) \\
X^G \arrow[u] \arrow[r] & \mathrm{Hom}(G,X) \arrow[u, "\Delta"'] \arrow[r, equal] & \mathrm{Hom}(G,X) \arrow[u, "\Delta_X \circ -"']
\end{tikzcd}$$

where each square is cartesian.

Deduce from the fact that X is separated, that $\mathrm{Hom}(G, X)$ is a closed subfunctor of $\mathrm{Hom}(G, X \times X)$. This can be reduced to the affine case, in which case it reduces to the fact that Weil restriction preserves closed subfunctors.

Now use the fact that the left square is cartesian that X^G is a closed subfunctor of X. But the closed subfunctors of a scheme are exactly the closed subschemes. □

Remark 4.1.2. The above statement can also be proved via the following stronger statement:

Claim: *Let S be a quasi-compact scheme. Let X be a smooth S-scheme where the geometric fibres are connected. Assume that there is a section $e : S \to X$. For every closed subscheme Y of X, the Weil restriction $\prod_{X/S} Y$ is represented by a closed subscheme T of S.*

To see this, let X_n the nth infinitesimal neighbourhood of e in X, and $Y_n = Y \cap X_n$. As X is smooth over S, X_n is finite locally free over S. Moreover, Y_n is a closed subsscheme of X_n, and hence

$$\prod_{X_n/S} Y_n \subset \prod_{X_n/S} X_n \cong S$$

is a closed subfunctor of S which is representable by a scheme T_n.

Note that $\prod_{X_n/S} Y_n = \prod_{X_n/S} Y_1$, because for an X_n-scheme Z $\mathrm{Hom}_{X_n}(Z, Y_n) \cong \mathrm{Hom}_{X_n}(Z, Y_1)$. On the other hand $T_{n+1} = \prod_{X_{n+1}/S} Y_1 \subset \prod_{X_n/S} Y_1 = T_n$ because morally, we restrict "further down".

Thus we obtain a decreasing sequence of closed subschemes of S

$$T_1 \supset T_2 \supset \cdots \supset T_n \supset \cdots$$

which becomes stationary as S is noetherian: set $T = T_n$ for n large enough.

$$\begin{tikzcd}
Y \arrow[rr, hook] \arrow[dd] & & X \arrow[dd] & \\
& Y_n \arrow[ul, hook] \arrow[rr, hook] \arrow[d] & & X_n \arrow[ul, hook] \\
T \arrow[r, hook] & T_n \arrow[r, hook] & S \arrow[ur] &
\end{tikzcd}$$

To show that T represents $\prod_{X/S} Y$, we have to show that over T, Y and X coincide. We know that over T we have $X_n = Y_n$ for all n.

Restrict to the affine case: let T be affine, $X = \mathrm{Sp}(A)$, let J be the ideal defining Y, and I the ideal defining the section e, i.e. $X_n = \mathrm{Sp}(A/I^{n+1})$ and $Y_n = \mathrm{Sp}(A/J + I^{n+1})$. The equality $Y_n = X_n$ implies $J \subset I^{n+1}$ hence $J \subset I^\infty$. To show $X = Y$, it suffices to show $I^\infty = 0$.

Note that A is noetherian, I^∞ a finite type A-module, such that $II^\infty = I$. Thus I^∞ is anihilated by an element of the form $1 - a$, $a \in I$. This can be seen as a function on X which is trivial on a irreducible component, but it is 1 on e. This is a contradiction and it follows $I^\infty = 0$.

With this we can deduce the above theorem:

Consider the morphism

$$G \times X \to X \times X, (g, x) \mapsto (x, gx).$$

Let Y be the preimage of the diagonal. The projection $G \times X \to X \cong \mathrm{Sp}(k) \times X$ has a section given by $e \times \mathrm{id}$. Thus $G \times X$ is an X-scheme with a section. $X^G \subset X$ is the locus where Y and $G \times X$ coincide. Thus $X^G \cong \prod_{G \times X/X} Y$ which is a closed k-subscheme of X by what we said above.

Remark 4.1.3. Note that X^G is not flat in general.

We can apply the above to the action of a k-group scheme on itself via conjugation.

Definition 4.1.4. Define the center of a smooth k-group scheme G.

4.2 Orbits

In this section, let k be a field.

Definition 4.2.1. Let G be a smooth k-group scheme which acts on a k-scheme X. For a k-point $x \in X(k)$, the orbit map is defined as follows:
Give a useful definition of the orbit map $\alpha_x : G \to X$.

Remark 4.2.2. In the situation above, we say that G acts transitively on X, if $G(\overline{k})$ acts transitively on $X(\overline{k})$ for an algebraic closure $\overline{k}$ of k. In this case, the orbit map is surjective for all $x \in X(k)$, because it is on all $\overline{k}$-points.

Denote by G_x the functor defined by

$$G_x : \mathrm{Alg}_k \to \mathrm{Ens}, R \mapsto \{g \in G(R) \mid gx = x\}.$$

Lemma 4.2.3. *If X is separated, G_x is a closed k-subgroup scheme of G.*

Proof. Argue that in this case, G_x is given as a fibre product

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] & X \times_k X \arrow[d] \\ G \arrow[r] & X \end{tikzcd}$$

for appropriate morphisms. □

Definition 4.2.4. Let G be a smooth k-group scheme which acts on a separated k-scheme X. For a k-point $x \in X(k)$, the subgroup scheme G_x of G is called the *isotropy group at* x.

Let $|O_x|$ be the image of the continuous map $|\alpha_x|$ of underlying topological spaces and denote by $\overline{|O_x|}$ is closure in $|X|$.

Proposition 4.2.5. *Let G be a smooth k-group scheme which acts on a separated k-scheme X and $x \in X(k)$.*

(i) The subset $|O_x|$ is open in $\overline{|O_x|}$.

(ii) Let $\overline{O}_x$ be the reduced scheme associated to $\overline{|O_x|}$, and O_x the open subscheme associated to $|O_x| \hookrightarrow \overline{|O_x|}$. Then the morphism α_x factors through O_x.

(iii) The morphism $G \to O_x$ induces an isomorphis between the sheaf associated to the presheaf

$$R \mapsto G(R)/G_x(R)$$

and the sheaf $R \mapsto O_x(R)$.

Proof. (i) Use Chevalley's constructibility theorem to show that $|O_x|$ contains a dens open of $\overline{|O_x|}$. Use this to show that every point in $|O_x|$ is contained in an open neighbourhood.

(ii) Recall that G is reduced and hence α_x factors through the reduced scheme $\overline{O}_x$.

(iii) Let $\pi_x : G \to O_x$ be the morphism obtained above. Show that π_x is faithfully flat. (Show that O_x is smooth, using that G acts transitively.)

For the fpqc-sheaf $\mathscr{Q}$ associated to the presheaf $R \mapsto G(R)/G_x(R)$. We have to show that the morphism of sheaves $\mathscr{Q}(R) \to O_x(R)$ is an isomorphism.

With what was said above, argue that π_x is an fpqc-cover of O_x. Consider the fibre product $G \times_{O_x} G$. For $R \in \mathrm{Alg}_k$, what are the R-points in $G \times_{O_x} G$ and how are they related to G_x?

Use that $\mathscr{Q}$ is an fpqc-sheaf to obtain an exact sequence

$$\mathscr{Q}(O_x) \to \mathscr{Q}(G) \rightrightarrows \mathscr{Q}(G \times_{O_x} G).$$

Let α be the section of $\mathscr{Q}(G)$ which is induced by the identity on G. Show that it comes from a section of $\mathscr{Q}$ over O_x, and hence induces an inverse of the morphism of sheaves $\mathscr{Q} \to O_x$.

□

Remark 4.2.6. In the above proof we used that any constructible subset Y of a Noetherian space X contains an open dense subset of its closure $\overline{Y}$. This is a well-known fact, a proof can be found in [3, AG.1.3].

Indeed, one can write Y as union $\bigcup_i L_i$ of locally losed sets, and hence $\overline{Y} = \bigcup_i \overline{L}_i$. If $\overline{Y}$ is irreducible, then $\overline{Y} = \overline{L}_i$ for some i and $L_i \subset Y$ is open dense in $\overline{L}_i$.

In general, let Y_j be the irreducible components of $Y = \bigcup_j Y_j$. As they are closed in Y, the Y_j are also constructible in X. For every j, the closure $\overline{Y}_j$ is also irreducible and by what we said above Y_j contains a dense open of $\overline{Y}_j$. Because the $\overline{Y}_j$ are the irreducible components of $\overline{Y}$, $Y = \bigcup_j Y_j$ containsa a dense open set in $\overline{Y}$.

Only in 2011, the statement was proved for general (not necessarily Noetherian spaces [1, Lem. 2.1].

Corollary 4.2.7. *In the sitation above, the sheaf associated to the presheaf*

$$R \mapsto G(R)/G_x(R)$$

is representable by a scheme.

Remark 4.2.8. There are more general statements about representability of quotients of group schemes, but the above statement can already be very useful.

Corollary 4.2.9. *Let G be a smooth k-group scheme of finite type acting on a separated k-scheme X of finite type. For every $x \in X(k)$ the orbit O_x is open in its closure. The complement $\overline{O}_x \backslash O_x$ consists of orbits of smaller dimension. In particular, there is at least one closed orbit.*

4.3 Affine group schemes and their representations

Definition 4.3.1. Let G be an affine k-group scheme. A *representation of* G is a homomorphism

$$\rho : G \to \mathrm{GL}\,(V)$$

where V is a locally free k-module. We say that ρ is *faithful* if it is injective for every k-algebra. We say that ρ is *finite* if V is of finite rank.

Remark 4.3.2. Recall the definition of $\mathrm{GL}\,(V)$.

Remark 4.3.3. A homomorphism of affine algebraic group schemes is a monomorphism if and only if it is a closed immersion. Indeed, every homomorphism of affine algebraic groups factors as an embedding after a quotient map. Injectivity implies that the quotient map is an isomorphism. The converse is clear.

Remark 4.3.4. Give an equivalent definition of a representation of G in terms of an "action of G on V".
Thus it makes sense to call V a G-module.

Remark 4.3.5. Let G be represented by a k-algebra A. Then a representation of G on V is equivalent to giving an A-comodule structure on V, that is a k-linear map $\rho : V \to V \otimes A$ such that

$$\begin{aligned}(\mathrm{id}_V \otimes \mu^*) \circ \rho &= (\rho \otimes \mathrm{id}_A) \circ \rho \\ (\mathrm{id}_V \otimes e^*) \circ \rho &= \mathrm{id}_V\end{aligned}$$

where μ^* is the comultiplication and e^* is the counit.
Explain this one-to-one correspondence.

Lemma 4.3.6. *Let k be a field and G an affine algebraic k-group scheme. Any action of G on an affine k-scheme X induces a representation.*

Proof. Consider an affine k-variety $X = \mathrm{Sp}(B)$ on which $G = \mathrm{Sp}(A)$ acts $\xi : G \times X \to X$ and the induced morphism of k-algebras

$$\xi^* : B \to B \otimes_k A.$$

For bases $\{b_\beta\}$ of B and $\{a_\alpha\}$ of A and an arbitrary element $b \in B$ write

$$\xi^*(b) = \sum_{i=1}^{n} c_i b_{\beta_i} \otimes a_{\alpha_i}$$

Let $V_b \subset B$ be the k-subvector space generated by the b_{β_i}.

We want to define a representation $G \to \mathrm{GL}\,(V_b)$. Explain how the abstract group $G(k)$ acts on the k-algebra B: for $g \in G(k)$

$$g(b) = \sum_{i=1}^{n} c_i g(a_{\alpha_i}) b_{\beta_i}.$$

Deduce that the set $\{g(b); | \, g \in G(k)\}$ belongs to the k-vector space $V_b \otimes A$, and that in particular $b \in V_b \otimes A$.

Next show that $\xi^*(V_b) \subset V_b \otimes_k A$. For this use the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\xi^*"] \arrow[d, "\xi^*"] & B \otimes A \arrow[d, "1\otimes\mu^*"] \\
B \otimes A \arrow[r, "\xi^*\otimes 1"] & B \otimes A \otimes A
\end{tikzcd}$$

where $\mu^* : A \to A \times A$ is the comultiplication. Thus we have

$$(\xi^* \otimes 1)\left(\sum_{i=1}^{n} c_i b_{\beta_i} \otimes a_{\alpha_i}\right) = (1 \otimes \mu^*)\left(\sum_{i=1}^{n} c_i b_{\beta_i} \otimes a_{\alpha_i}\right)$$

Develop $\mu^*(a_\alpha)$ and compare above the right-hand side and left-hand side to show that $\xi^*(b_{\beta_i}) \in V_b \otimes A$.

Deduce that this gives a representation $\rho_b : G \to \mathrm{GL}\,(V_b)$. □

Proposition 4.3.7. *Let k be a field and G an affine algebraic k-group scheme. Then G has a faithful representation.*

Proof. For $x \in G(R)$ let $\tau_x : G(R) \to G(R)$ be defined by $\tau_x(y)(f) = (yx)(f)$. This is equivalent to a map $\tau_x^* : A \otimes R \to A \otimes R$ and is called the action by translation. Let A be generated as k-algebra by $f := \{f_1, \dots, f_n\}$ and V_f be the k-vectorspace generated by f.

As above it follows that V_f is stable under the G-action by translation and that it induces a morphism $\rho_f : G \to \mathrm{GL}\,(V_f)$

Show that the induced morphism of k-algebras $B \to A$ where B represents $\mathrm{GL}\,(V_f)$ is surjective. □

Definition 4.3.8. Let k be a field and G an affine algebraic k-group scheme. The representation induced by multiplication

$$\mu : G \times G \to G$$

described in the proof above is called the regular representation.

Corollary 4.3.9. *Let k be a field. Every affine algebraic k-group scheme G is a closed subgroup of a $\mathrm{GL}\,(V)$, where V is a finite dimensional k-vector space.*

Proposition 4.3.10. *Let k be a field, $G = \mathrm{Sp}(A)$ an affine k-group scheme with a closed subgroup scheme H defined by the ideal I. Then for any k-algebra R*

$$H(R) = \{g \in G(R) \mid \tau_g^*(I \otimes R) = I \otimes R\}.$$

Proof. Use that the translation τ_g^* is given on $G(R)$ by translation. Argue that for $g \in G(R)$ with $\tau_g^*(I \otimes R) = I \otimes R$ and $a \in I$ one has $g(a) = \tau_g(e_G)(a) = 0$ as $\tau_g^*(a) \in I \otimes R$. □

The following is called Chevalley's semi-invariant theorem.

Theorem 4.3.11. *Let G be a smoothaffine k-group scheme of finite type and H a sbgroup scheme of G. Then there is a representation*

$$\rho : G \to \mathrm{GL}(V)$$

where V is a finite dimensional k-vector space, and a line $L \subset V$, such that

$$H = \{g \in G \mid \rho(g)L = L\}.$$

Proof. Let $G = \mathrm{Sp}(A)$ and I be the ideal defining H. Argue similar to above that there is a finite dimensional k-subvectorspace W of A containing generators of I as an ideal and stable under G-action (by multiplication/translation).
This gives an action $\rho : G \to \mathrm{GL}(W)$
Let $M = W \cap I$. Describe H in terms of M.
To obtain a vectorspace with a one-dimensional subspace $L \subset V$ take exterior powers. □

Remark 4.3.12. In other words every algebraic subgroup of an algebraic group arises as the stabilizer of a one-dimensional subspace in a finite dimensional representation.

Remark 4.3.13. The above means that $H(R)$ is the stabiliser of L_R in V_R. Applying this to $R = k[\varepsilon]$ with $\varepsilon^2 = 0$, we find that

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid d\rho(x)L \subset L\}$$

where $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{g} = \mathrm{Lie}(G)$.

Corollary 4.3.14. *Let G be an affine smooth k-group scheme, H a subgroup scheme of G. Then the sheaf associated to the presheaf $R \mapsto G(R)/H(R)$ is represented by a quasi-projective k-scheme.*

Proof. Note that G acts on the lines in V and hence on a projective space, where L is an element. Then use Proposition 4.2.5 (iii) to identify the quotient in question with the orbit of L. □

Let H be an affine k-group scheme of finite type. A character of H is a homomorphism

$$\chi : H \to \mathbb{G}_m.$$

Let $X(H)$ be the abelian group of characters of H and $C_H := \mathrm{Hom}(X(H), \mathbb{G}_m)$. As a scheme

$$C_H = \mathrm{Sp}(k[X(H)]).$$

There is a surjective morphism

$$\rho : H \to C_H$$

and hend an injective morphism of k-algebras $k[X(H)] \to A_H$.

Theorem 4.3.15. *Let G be an affine k-group scheme H a normal subgroup scheme. Then there is a finite representation $\rho : G \to \mathrm{GL}\,(W)$, such that H is the kernel of ρ. In particular the quotient G/H is an affine k-group scheme.*

Proof. Apply Chevalley's semi-invariant theorem to obtain a representation $G \to \mathrm{GL}\,(V)$ and a line $L \subset V$ such that H is the stabiliser of L. Argue that H acts on L by a character $\chi : H \to \mathbb{G}_m$, that is the representation $H \to \mathrm{GL}_L$ factors through χ.

Consider the sum of all lines stabilised by H. As G normalises H, this sum is also a representation of G and we can replace V by this subspace and ths assume that V is generated by H-stable subspaces. Moreover, they have to be linearly independent.

Let W be the subspace of $\mathrm{End}(V)$ which stabilises all the lines L. $\mathrm{GL}\,(V)$ acts on $\mathrm{End}(V)$ by adjoint action and this action leaves W stable. Thus we obtain a representation $\rho : G \to \mathrm{GL}\,(W)$ with kernel H. $\square$

5 Chevalley's structure theorem

5.1 Abelian varieties

Definition 5.1.1. For a commutative ring with unit k, an **abelian k-scheme** is a smooth proper k-group scheme with connected geometric fibres. If k is a field this is called an **abelian k-variety**.

Example 5.1.2. Give an example for an abelian k-scheme.

Remark 5.1.3. Let k be a field. As an algebraic k-group scheme, an abelian k-variety is quasi-projective, and hence projective.

Lemma 5.1.4. *Let k be a field and A an abelian k-variety. Then the group law of A is commutative.*

Proof. It suffices to assume that k is algebraically closed. Consider the diagram

$$\begin{tikzcd} A \times_k A \arrow[rr, "h"] \arrow[dr, "\mathrm{pr}_1"'] & & A \times_k A \arrow[dl, "\mathrm{pr}_1"] \\ & A & \end{tikzcd}$$

where h is given by $(x, y) \mapsto (x, xyx^{-1}y^{-1})$. Argue that the fiber of h over $e \in A$ is constant with value (e, e). This mus be true for an open neighbourhood U of e as well. □

Proposition 5.1.5. *Let k be a field.*

(i) Any morphism from an abelian k-variety to an affine algebraic k-group (i.e. an affine k-gropu scheme of finite type) is trivial.

(ii) Any morphism from an affine smooth connected algebraic k-group to an abelian k-variety is trivial.

Proof. Show first that for a proper reduced connected k-scheme $\Gamma(X, \mathscr{O}_X) = \mathbb{G}_a(X)$ is a field: a morphism $X \to \mathbb{G}_a$ must have finite image, then use connectedness to show it is constant. Deduce from this the first statement.

For the second statement let $G \to A$ be a morphism from an affine smooth connected algebraic k-group to an abelian k-variety with kernel H. Consider the sheaf associated to the presheaf $R \mapsto G(R)/H(R)$. On the one hand it can be identified with the image of G in A, on the other hand it is an affin smooth connected k-group scheme. Argue that it can be identified with a closed sub-group scheme of A which is affine, connected and reduced. Hence it must be trivial. □

5.2 The adjoint group

Let k be a field. We will need the following statements:

(i) Every monomorphism of algebraic groups is a closed immersion.

(ii) Let N be a normal subgroup of an algebraic group G. The homomorphism of fpqc-sheaves

$$G \to \widetilde{G/N}$$

is represented by a faithfully flat homomorphism of algebraic groups. In particular, the fpqc-sheafifcation of the presheaf $R \mapsto G(R)/N(R)$ is represented by an algebraic group denoted G/N.

Lemma 5.2.1. *Let G be an algebraic k-group scheme, X a connected separable algebraic scheme and $G \times X \to X$ a faithful action. If there is a fixed point P, the G is affine.*

Proof. Argue that G acts on the local ring $\mathscr{O}_{X,P}$. Let $\mathfrak{m}_p$ be its maximal ideal. Forming the quotient $\mathscr{O}_{X,P}/\mathfrak{m}_P^{n+1}$ commutes with extension of the base. Thus for any k-algebra R, there is a natural morphism

$$G(R) \to \mathrm{Aut}(R \otimes_k (\mathscr{O}_{X,P}/\mathfrak{m}_P^{n+1})),$$

and hence a representation

$$\rho_n : G \to \mathrm{GL}\,(\mathscr{O}_{X,P}/\mathfrak{m}_P^{n+1}).$$

Let K_n be its kernel. Argue that the descending sequence of subgroups

$$G \supset K_0 \supset \cdots \supset K_n \supset K_{n+1} \supset \cdots$$

becomes stationary: $K = \bigcap K_n = K_{n_0}$.

Let $\mathcal{I}$ be the ideal sheaf defining the closed subscheme $X^K \subset X$. Argue that X^H contains an open neighbourhood of P as $\mathcal{I}\mathscr{O}_{X,P} \subset \bigcup \mathfrak{m}_P^n = 0$.

As X^K is closed and X connected $X^K = X$, and $K = e$. Thus ρ_n is injective for n big enough, hence a closed immersion (every monomorphism of algebraic groups is a closed immersion), hence G is affine. □

Let G be an algebraic k-group scheme. The action of G on itself by conjugation defines a representation of G on the k-vector space $\mathscr{O}_{G,e}/\mathfrak{m}_e^{n+1}$

$$\rho_n : G \to \mathrm{GL}\,(\mathscr{O}_{G,e}/\mathfrak{m}_e^{n+1}).$$

Proposition 5.2.2. *Let G be a connected algebraic k-group scheme.*

(i) For sufficiently large n, the kernel of this representation ρ_n is the centre of G.

(ii) The fpqc*-sheaf associated to the presheaf $R \mapsto G(R)/ZG(R)$ is represented by an affine k-group schem.*

Proof. Apply the above lemma to the faithfu action $G/ZG \times G \to G$. □

Corollary 5.2.3. *Every abelian subvariety A of a connected algebraic k-group scheme G is contained in the centre of G In particular, every abelian k-variety is commutative.*

Proof. Apply Proposition 5.1.5. □

Definition 5.2.4. Let G be a connected algebraic k-group scheme. The group scheme representing the fpqc-sheaf associated to the presheaf $R \mapsto G(R)/ZG(R)$ is called the **adjoint group** of G.

5.3 A rigidity lemma

Lemma 5.3.1. *Let X be a proper reduced and connected scheme over k, let Y and Z be k-schemes and $\phi : X \times Y \to Z$ a morphism such that there is $y_0 \in Y(k)$ and $z_0 \in Z(k)$ with $\phi(X \times \{y_0\}) = z_0$. For a point $x_0 \in X(k)$ define a morphism $\psi : Y \to Z$ by $\psi(y) = \phi(x_0, y)$. Then $\phi(x, y) = \psi(y)$.*

Proof. It suffices to show this on an open dense of $X \times Y$. Let Z' be an affine open neighbourhood of z_0 in Z. As X is proper, the projection $\operatorname{pr}_Y : X \times Y \to Y$ is closed. Hence $\operatorname{pr}_Y(\phi^{-1}(Z \backslash Z')) \subset Y$ is closed.
Identify the complement

$$Y' = Y \backslash \operatorname{pr}_Y(\phi^{-1}(Z \backslash Z'))$$

and show that it is not empty.
Argue that for every $y \in Y'$ the restriction of ϕ to $X \times \{y\}$ has image in the open affine Z' and hence is constant $\phi(x, y) = \phi(x_0, y)$ for every x. Deduce the statement. □

Proposition 5.3.2. *Let A be an abelian k-variety and G an algebraic k-group scheme. Then every morphism of k-schemes $f : A \to G$ with $f(e_A) = e_G$ is a homomorphism of k-gropu schemes.*

Proof. Consider the morphism

$$\phi : A \times_k A \to G$$
$$\phi(x, y) = f(xy) f(x)^{-1} f(y)^{-1}.$$

Show that

$$\phi(A \times \{e_a\}) = \phi(\{e_A\} \times A) = e_G$$

and apply the rigidity lemma. □

Remark 5.3.3. Apply the proposition to the morphism $A \to A, a \mapsto a^{-1}$ to give another proof that A is commutative.

Remark 5.3.4. Let X be a proper integral k-scheme and $e \in X(k)$ a point. Let $m : X \times X \to X$ be a morphism such that

$$m(x, e) = m(e, x) = x$$

for all x. Then X is an abelian k-variety with group law given by m.

5.4 Extension of rational morphisms

For simplicity we assume here that k is a field. However, we might extend the results easily to any normal noetherian base scheme.

Definition 5.4.1. Let X, Y be k-schemes. A rational morphism $X \overset{f}{\dashrightarrow} Y$ is the data of a dense open $U \subset X$ together with a morphism $f : U \to Y$. Two pairs (U, f) and (U', f) are equivalent, if and only if their restrictions to $U \cap U'$ coincide. In this case f is defined on $U \cup U'$, and there is a maximal open subset of X where the rational morphism is a morphism.

Example 5.4.2. Give an example of a rational map that does not extend.

In this section, we would like to obtain the following statement which was discovered by Weil:

Theorem 5.4.3. *Every rational morphism $X \overset{f}{\dashrightarrow} A$ from a smooth k-scheme to an abelian scheme is defined everywhere.*

There are several ways to prove this.

Lemma 5.4.4. *A rational map $X \overset{f}{\dashrightarrow} Y$ from a normal variety to a complete variety is defined on an open subset $U \subset X$ whose complement has codimension $\geqslant 2$.*

Proof. Consider first the case of curves and then reduce to this case. Compare [9, Thm. 3.1]. □

Lemma 5.4.5. *Let $X \overset{f}{\dashrightarrow} G$ be a rational map from a non-singular variety to a group variety. Then either it is defined everywhere or the points where it is not defined form a close subset of pure 1.*

Proof. Give a sketch of this Compare [9, Lem. 3.3]. □

Proof of Theorem 5.4.3. Combine the previous lemmata to show the theorem. □

5.5 Another rigidity lemma

Lemma 5.5.1. *Let V and W be smooth separated k-varieties of finite type. Let $\phi : V \times W \to A$ be a morphism into an abelian variety such that there is $w_0 \in W(k)$ and $a_0 \in A(k)$ with $\phi(V \times \{w_0\}) = a_0$. Let $v_0 \in V(k)$ and $\psi : W \to A$ be the morphism defined by $\psi(w) = \phi(v_0, w)$. Then $\phi(v, w) = \psi(w)$.*

Sketch of proof. Let $v \in V$ be another point in V. It is known that there is an irreducible curve C on V passing through v and v_0. After normalising, we may assume that it is smooth. Thus we may assum that V is a curve. Let $\overline{V}$ be its compactification. Then be the above extension theorem the morphism

$$V \times W \to A$$

extends to $\overline{V} \times W$. Then one can reduce to the first rigidity lemma. □

Proposition 5.5.2. *Let G be an algebraic k-group and A an abelian k-variety. Let $f : G \to A$ be a morphism (of schemes) with $f(e_G) = e_A$. Then f is a homomorphism (of group schemes).*

Proof. Consider the morphism

$$\phi : G \times G \to A, \phi(x, y) = f(xy)f(x)^{-1}f(y)^{-1},$$

which satisfies $\phi(G \times \{e_G\}) = \phi(\{e_G\} \times G) = \{e_A\}$. Then the second rigidiy lemma implies that ϕ is trivial. □

5.6 Proper and non-proper group schemes

The following follows Rosenlicht's [11] argument.

Definition 5.6.1. A morphism of algebraic groups is calles an isogeny, if it is surjective with finite kernel.

Proposition 5.6.2. *Let G be an algebraic k-group and A a subgroup scheme of G which is an abelian variety. Then there is a normal subgroup scheme N of G such that the map*

$$A \times N \to G, (a, g) \mapsto ag$$

is an isogeny. In particular G is proper if and only G_1 is.

Proof. According to [7, Exp. 6, Thm. 3.2] the quotient G/A is representable by a scheme Q. Let $\pi : G \to Q$ be the porjection, which is smooth and projective. Let $\eta_Q = \mathrm{Sp}(K)$ be its generic point. Its preimage under π is a principal homogeous space over $A \otimes_k K$, in particular R is smooth over K.

By [4, 2.2, Cor.frm[o]–3], the closed points of R whose residue field is a finite separable extension of K are dense in R. Let α be such a closed point, denote by $\kappa(\alpha)$ is residue field, let n be the degree of teh extension $\kappa(\alpha)/K$.

Then

$$L \otimes_K \overline{K} = \overline{K} \times \cdots \times \overline{K}$$

n-times. Let $\overline{\alpha}_1, \dots, \overline{\alpha}_n$ be the $\overline{K}$-points over α. Let $\widetilde{\kappa(\alpha)}$ be the Galois closure of $\kappa(\alpha)$ in $\overline{K}$, and $\Gamma = \mathrm{Gal}\left(\widetilde{\kappa(\alpha)}/K\right)$. The points $\overline{\alpha}_i$ are already defined over $\widetilde{\kappa(\alpha)}$.

There is a homomorphism

$$R \otimes_K \widetilde{\kappa(\alpha)} \to A \otimes_k \widetilde{\kappa(\alpha)}, r \mapsto \sum_{i=1}^{n} (r - \alpha_i)$$

where $a_i := r - \alpha_i$ is the unique point $a_i \in A$ such that $a_i + \alpha_i = r$.

This morphism is Γ-equivariant, and hence descneds to a morphism $b : R \to A \otimes_k K$ with

$$b(a + r) = na + b(r)$$

By Theorem 5.4.3 is extends to a morphism

$$b : G \to Q \times A$$

and hence a morphism $G \to A$ with $b(a + g) = na + b(g)$

After translation, one may assu $b(e_G) = e_A$, hence it is a morphism of group schemes. The restriction to A is the multiplicaiton by n

$$[n] : A \to A.$$

Let G_1 be the kernel of $b : G \to A$, which is a normal subgroup. The intersection $G \cap A$ is the kernel of multiplicaiont by n in A, and hece a finite k-group scheme.

As A is contained in the centre of G, the map

$$A \times N \to G, (a, g) \mapsto ag$$

is a morphism of group schemes.

Show that it is surjective.

□

Lemma 5.6.3. *Let G be an algebraic group over an algebraically closed field k. If G is not proper, it contains an affine sub-group scheme of positive dimension.*

Proof. Show this if you are motivated, or find a reference. □

5.7 Chevalley's theorem

The proof of Chevalley's theorem that we discuss here is the one due to Rosenlicht [11].

Theorem 5.7.1. *Let G be an algebraic connected group of finite type over a field k. There is a normal affine connected subgroup L of G such that $A := G/L$ is an abelian variety. Moreover, L contains all affine normal connected subgroups of G.*

Proof. Let C be the centre of G. If C is trivial, argue that G is isomorphic to its adjoin group and conclude.

If C is proper, there is a quasi-complement G_1, $\dim G_1 < \dim G$, and an isogeny $C \times G_1 \to G$. Argue thatIf one takes an affine normal linear connected subgroup $L \subset G_1$, such that G_1/L is an abelian variety the same is true for G/L.

Thus assume that C is not proper. Then by the previous lemma, there is an affine connected subgroup L_1 of positive dimension which is therefore an affine normal connected subgroup of G. Thus one can argue inductively via G/L_1 using the fact that he extension of an affine group scheme by an affine group scheme is affine by the subsequent lemma. □

Lemma 5.7.2. *Assume that there is an exact sequence*

$$1 \to l \to G \to G_1 \to 1$$

of smooth group schemes such that L and G_1 are affine. Then G is affine as well.

Proof. Argue that $G \to G_1$ is faithfully flat. By basechaggne from G_1 to G one has

$$G \times_{G_1} G = G \times L.$$

Thus the morphism $G \times_{G_1} G \to G$ is affine. By faithfully flat descent, $G \to G_1$ is affine, that is the preimage of an affine open is affine. Deduce that G is affine. $\square$

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