k-analytic spaces

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As in the previous parts, this is taken from [1, Section 4.4 and 5.1] In this talk, we want to explain how to use k-affinoid spaces in the sense of Berkovich to globalise this notion and define k-analytic spaces. Let k be a non-archimedean normed field, $\mathscr{M}(\mathscr{A})$ the spectrum of a (commutative) k-Banach algebra. This can be endowed with a topology that makes it a compact Hausdorff space. Analogously to rigid spaces, we would like to see this as local building blocks to construct some sort of ringed space. However, the topology, which is contrary to rigid spaces an actual topology, creates some obstructions as we will see later. Let's see what happens, when we try to mimick the rigid approach.

1 Affinoid subdomains

We define an analogue of affinoid subdomain of rigid geometry using as well a universal mapping property.

Definition 1.1. Let $\mathscr{A}$ be a k-affinoid algebra. A subset $U \subseteq \mathscr{M}(\mathscr{A})$ is called a k-affinoid subdomain, if there is a bounded map $i : \mathscr{A} \to \mathscr{A}'$ of k-affinoid algebras such that:

- the imasge of the "dual" map $\mathscr{M}(i)$ is contained in U
- it is universal in the sense that any bounded map $\phi : \mathscr{A} \to \mathscr{B}$ such that the dual map $\mathscr{M}(\phi)$ has image in U, factors uniquely through i:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\phi} & \mathscr{B} \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \exists!} & \\ & \mathscr{A}' & \end{array}$$

This determines $\mathscr{A}'$ uniquely up to unique bounded isomorphism, thus we write $\mathscr{A}_U$ and call it the coordinate ring of U.

There is in fact a stronger universal property which can be proved using the k-analytic analogue of the Gerritzen–Grauert Theorem: let $U = \mathscr{M}(\mathscr{A}_U) \subset \mathscr{M}(\mathscr{A})$ be an affinoid subdomain, choose a point $x \in U$. Then the completed residue fields of x for $\mathscr{A}_U$ and $\mathscr{A}$ coincide, in other words, if $\kappa(x)$ is the completion of the fraction field of $\mathscr{A}\,/\ker(|-|_x)$, then projection map $\mathscr{A} \to \kappa(x)$ factors uniquely through the canonical map $\mathscr{A} \to \mathscr{A}_U$. Indeed, affinoid subdomains satisfy a universal property with respect to arbitrary k-Banach algebras (and not only k-affinoid algebras). This stronger universal property follows as mentioned before from the Gerritzen–Grauert theorem for k-analytic spaces.

It follows also, that the map $\mathscr{M}(\mathscr{A}_U) \to U$ from the definition is a homeomorphism. Moreover, the concept of affinoid domains is wel-behaved with respect to (any analytic) base change (using completed tensor product). Consequently, K-valued points for K/k an analytic extension don't pose a pronblem anmore as they did in the rigid setup. Also using completed tensor product, one can show, that intersections of affinoid domains are again affinoid. On the other hand it is much harder to show that the algebras $\mathscr{A}_U$ are $\mathscr{A}$-flat.

As in the rigid case, it is quite straight forward to define Weierstraß, Laurent and rational domains. Comparing these with the open sets, that generate the topology on $\mathscr{M}(\mathscr{A})$, it becomes clear, that in general affinoid subdomains cannot be open in this topology. Hence the slogan: "open sets in rigid geometry are closed sets in Berkovich theory".

2 Sheaves

As we want to construct coherent sheaves on affinoid spaces and as well globalise this notion, we take note, that the theorems of Tate and Kiehl from rigid geometry carry over to the new setting. They considered the strict case (i.e. the r_i in the definition of affinoid algebra are equal to 1). However, instead of scaling with r_i's one can consider appropriate analytic extensions of the base field, which in this setting is perfectly fine. The main result then reads as follows.

Theorem 2.1. *Let $\mathscr{A}$ be a k-affinoid algebra, and M a finite $\mathscr{A}$-module. Let $\mathfrak{U}$ be a finite collection of affinoid subdomains of $X = \mathscr{M}(\mathscr{A})$ that cover X. Define $M_U := \mathscr{A}_U \otimes_{\mathscr{A}} M$ for any affinoid subdomain $U \subset X$. Then the Čech complex $C^\bullet(\mathfrak{U}, M)$ is an exact sequence.*

The issue of a structure sheaf will be addressed in the next section, that deals with globalisation.

3 Globalisation

It is not obvious how to construct a structure sheaf. There seems to be no canonical way to associate rings to the open sets that generate the topology of an affinoid space $\mathscr{M}(\mathscr{A})$. The more natural thing is to work with affinoid subdomains, which are in general closed. In order to fix this, one has to generalise the notion of topology, using so called quasi-nets.

Definition 3.1. A quasi-net on a locally Hausdorff space X is a collection τ of compact Hausdorff subsets $V \subset X$ such that each $x \in X$ has a neighbourhood of the form $\bigcup V_j$ for finitely many $V_j \in \tau$ with $x \in \bigcap V_j$.

The local finiteneyy condition is reminiscent of the finiteness condition in a Grothendieck topology, and is th key reason, why they can provide a workable substitute for usual open coverings. We procede to replace the framework of ringed spaces by k-affinoid atlases, which is inspired by the set of all affinoid subdomains on a k-affinoid space.

Definition 3.2. Let X ba a locally Hausdorff space. A k-affinoid atlas on X is th data of a quasi-net τ on X such that

- for all $U, U' \in \tau$, the set $\{V \in \tau \mid V \subseteq U \cap U'\}$ is a quasi-net on $U \cap U'$.
- For each $V \in \tau$, there is assigned a k-affinoid algebra $\mathscr{A}_V$ such that $\mathscr{M}(\mathscr{A}_V) \cong V$ is a homeomorphism amd such that if $V' \in \tau$ is a subset of V it can be expressed as a k-affinoid subdomain of $\mathscr{M}(\mathscr{A}_V)$. The collection of this data is denoted by $\mathscr{A}$.

The triple $(X, \mathscr{A}, \tau)$ is called a k-analytic space.

Consider for example an affinoid space $\mathscr{M}(\mathscr{A})$. One choice of quasi-net τ would be the set of all affinoid subsets. This clearly fulfilles the requirements of the definition.However, consider instead the more courious case of $\tau' = \{X\}$. This is also a quasi-net, and both choices together with the associated affinoids form k-analytic spaces. Clearly, we want to consider the two analytic spaces to be naturally isomorphic. It seems that the latter one contains less information, however, we will see that it is possible to recover the information of the space from both quasi-nets as we will see.

In order to make this precise, we need to understand the concept of morphism of k-analytic spaces. However, as this is quite subtle considering the fact that we cannot pass to the sheaf-language, we weill describe only a certain type of morphism, that is needed for the construction of some sort of universal or maximal quasi-net.

Definition 3.3. Let $(X, \mathscr{A}, \tau)$ an $(X', \mathscr{A}', \tau')$ be k-analytic spaces. A strong morphism $(X, \mathscr{A}, \tau) \to (X', \mathscr{A}', \tau')$ consists of a continuous map $\phi : X' \to X$ such that for all $V' \in \tau'$ and $V \in \tau$ with $V' \subseteq \phi^{-1}(V)$ there is a compatible k-Banach algebra map $\mathscr{A}_V \to \mathscr{A}_{V'}$, transitive in the pair (V, V').

Now lets start with a k-analytic space $(X, \mathscr{A}, \tau)$. In two steps we will arrive at a maximal k-analytic space. First enlarg τ, by letting $\overline{\tau}$ be the set of $V \subset X$ such that V is a k-affinoid subdomain in $\mathscr{M}(\mathscr{A}_{V'}) = V'$ for some $V' \in \tau$. In other words, we add all affinoid subdomains of all elements of τ. Aplying this to the example above of the affinoid case, one would get from the trivial quasi-net $\{X\}$ on X to the set of all affinoid subdomains of X. By choice of the elements of $\overline{\tau}$ there is a unique atlas structure $(X, \overline{\mathscr{A}}, \overline{\tau})$ (up to unique isomorphism).

But we are not done here. Let $\widehat{\tau}$ be the collection of subsets $W \subset X$ with a finite cover of $W_i \in \overline{\tau}$ such that

1. $W_i \cap W_j \in \overline{\tau}$
2. the natural map $\overline{\mathscr{A}}_{W_i}\widehat{\otimes}_k\overline{\mathscr{A}}_{W_j} \to \overline{\mathscr{A}}_{W_i\cap W_j}$ is surjective, such that the residue norm is equivalent to the given norm on the target (which is automatically satisfied for non-trivial absolute value ojn k.
3. denote by $\widehat{\mathscr{A}}_{\{W_i\}}$ the equaliser of

$$\prod \overline{\mathscr{A}}_{W_i} \rightrightarrows \prod \overline{\mathscr{A}}_{W_i\cap W_j}$$

   Then we require that $\widehat{\mathscr{A}}_{\{W_i\}}$ be affinoid and the canonical map $W \to \mathscr{M}(\widehat{\mathscr{A}}_{\{W_i\}})$ is a homeomorphism identifying each $W_i \subset W$ with a k-affinoid subdomain.

It can be shown, that the algebra $\widehat{\mathscr{A}}_{\{W_i\}}$ is independent of the choice of covering of W, so instead we write $\widehat{\mathscr{A}}_W$. If W was already in $\overline{\tau}$, then $\widehat{\mathscr{A}}_W = \overline{\mathscr{A}}_W$. The assignement $W \mapsto \widehat{\mathscr{A}}_W$ for all $W \in \widehat{\tau}$ provides a k-affinoid atlas, and $(X, \widehat{\mathscr{A}}, \widehat{\tau})$ is a k-analytic space.

It is maximal in the sense that doing the same procedure again will give back the same structure, i.e. $(X, \widehat{\widehat{\mathscr{A}}}, \widehat{\widehat{\tau}}) = (X, \widehat{\mathscr{A}}, \widehat{\tau})$.

The canonical morphism $(X, \mathscr{A}, \tau) \to (X, \widehat{\mathscr{A}}, \widehat{\tau})$ is a string morphism in the sense above. Now invert these specific strong morphisms formally. Thus we localise the category of k-analytic spaces with respect to these morphisms, which identifies a maximal atlas with all the atlases that induces it.

One can do this over, with only using strict k-analytic spaces, which is the natural category when promoting a rigid space to a k-analytic space. In a similar thought, it is possible to consider the category of all K-analytic spaces for all analytic extensions K/k in one category. This remedies the question how to deal with K-valued points, that posed a problem in the setting of rigid spaces. Moreover it is a recent result by Temkin, that strictly k-analytic spaces form a full subcategory of k-analytic spaces.

Originally, Berkocich considered k-analytic spaces, where each point has an affinoid neihbourhood. But this set of spaces is rather small, not including cases that come from rigid geometry. Consider the following light generalisation.

Definition 3.4. A k-analytic space $(X, \mathscr{A}, \tau)$ is called good, if every $x \in X$ has a neighbourhood $V \in \widehat{\tau}$.

There is a natural analytification functor. The k-analytic spaces obtained from algebraic k-schemes are always good. However, there are k-analytic spaces associated to a rigid space that are not good.

To define sheaves, we need one more definition to replace the term "admissible open".

Definition 3.5. Let $(X, \mathscr{A}, \tau)$ be a k-analytic space. A k-analytic domain in X is a subset $Y \subset X$ such that for all $y \in Y$ there exist $V_1, \ldots, V_n \in \widehat{\tau}$ with $y \in \bigcap V_j$ and $\bigcup V_j$ a neighbourhood of y in Y (in particular all $V_i \subset Y$.

References

[1] Brian Conrad. Several approaches to non-archimedean geometry. In *p-adic Geometry*, volume 78 of *AMS University Lecture Series*. Amer. Math. Soc., Providence, RI 41, 2008. Lectures from the 2007 Arizona Winter School.