Constructions on k-analytic spaces

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We continue our journey to explore non-archimedean geometry, finishing up with a little deeper study of Berkovich spaces. This part will cover (but not follow one-to-one) Sections 5.2 and 5.3 inlcuding the section we skipped and need now, Section 4.5 of [1].

To be more specific, we will discuss some constructions from Berkovich, (also called k-analytic) geometry, some of whose show what makes the theory so appealing, or powerful, and some make a relation to rigid analytic geometry possible.

1 Fibre products

As we have seen in the previous part, one can globalise affinoid spaces, by glueing along so-called k-analytic domains. This proces can be taken one step further, and one can patch together in the same manner k-analytic spaces and morphisms. One of the constructions that can be done in this way is the globalisation of fibre products.

Let k be a non-archimedean field. As we have seen in rigid geometry, there are several different versions of the tenorproduct. For a pair of morphisms of k-affinoid algebras

$$\mathscr{A} \to \mathscr{A}' \text{ and } \mathscr{A} \to \mathscr{A}''$$

we obtain via the completed tensor product over $\mathscr{A}$ another k-affinoid algebra $\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}''$. Then the induced morphisms of k-analytic spaces

$$\mathscr{M}(\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'') \to \mathscr{M}(\mathscr{A}') \text{ and } \mathscr{M}(\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'') \to \mathscr{M}(\mathscr{A}'')$$

have the same composition to $\mathscr{M}(\mathscr{A})$ so that the diagramm

$$\begin{array}{ccc} \mathscr{M}(\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'') & \longrightarrow & \mathscr{M}(\mathscr{A}'') \\ \downarrow & & \downarrow \\ \mathscr{M}(\mathscr{A}') & \longrightarrow & \mathscr{M}(\mathscr{A}) \end{array}$$

commutes, and by the universalt property of the tensor product and of affinoid subdomains, $\mathscr{M}(\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'')$ satisfies the universal property of the fiber product in the category of k-analytic spaces, so that we may set

$$\mathscr{M}(\mathscr{A}') \times_{\mathscr{M}(\mathscr{A})} \mathscr{M}(\mathscr{A}'') := \mathscr{M}(\mathscr{A}' \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'').$$

It is also important in the proof to note that one can recover uniquely morphisms on k-analytic spaces frim a specification of the morphism on a suitable affinoid cover. This latter fact also allows us to globalise this construction uniquely to obtain fibre products of arbitrary k-analytic spaces.

We don't go into details here, but make some remarks that should bring the construction in perspective. Because of the nature of quasi-nets that are used in tis globalisation the glueing process is a bit more invoilved as in the case of schemes, althogh it follows a similar strategy. Because of the nature of the construction, fibre products are well behaved with respect to k.analytic domains, meaning that the fiber product of two k-analytic domains over a k-analytic domain, is a k-analytic domain itself.

The other point to note is that the underlying topological space of the fiber product of two k-analytic spaces $|X' \times_X X''|$ is not the same as the fibre product of the underlying topological spaces of each k-analytic space in the category of topological spaces, although the natural map

$$|X' \times_X x''| \to |X'| \times_{|X|} |X''|$$

is a proper surjection. Note hereto, that the glueing proces for k-analytic spaces unlike the glueing process for topological spaces does not use open coverings, but quasi-nets.

Whle we are at it, recall that a morphism of topological spaces is called proper, if it is separated (the diagonal is a closed embedding) and universally closed.

Lastly, a point in $X' \times_X X''$ does not necessarily have a base of neighourhoods of the type $Y' \times_Y Y''$ for k-analytic domains, Y, Y' and Y''.

Question. Can you find an example for this?

2 Extension of the base field

Strictly speaking, this should be part of the previous section, as it a special case of fibre product. Let K/k ba an analytic extension, i.e. with compatible absolute value with respect to which it is complete. Then for a k-affinoid algebra $\mathscr{A}$ we obtain K-affinoid algebra

$$\mathscr{A}_K = K \widehat{\otimes}_k \mathscr{A}$$

and a natural morphism from $\mathscr{A}$ to $\mathscr{A}_K$. The induced morphism

$$\mathscr{M}(\mathscr{A}_K) \to \mathscr{M}(\mathscr{A})$$

from a K-affinoid space to a k-affinoid space serves as a fibre product

$$\mathscr{M}(\mathscr{A}) \times_{\mathscr{M}(k)} \mathscr{M}(K) := \mathscr{M}(\mathscr{A}_K)$$

in the category of analytic spaces over arbitrary analytic extension fields of k. This extension of the ground field functor is naturally transitiv with respect to further extensions of the base field.

We noted before that this is one advantage of k-analytic spaces over rigid analytic spaces, as it allows us to "see" points with coefficients in analytic extension fields of our base field.

The remarks made above for fibre products apply here as well.

3 Separated morphism

Similarly to the rigid analytic case, one can define closed immersions between k-analytic spaces. Thus it should be something along the lines:

Definition 3.1. A map between k-analytic spaces $f : (X', \mathscr{A}', \tau') \to (X, \mathscr{A}, \tau)$ is a closed immersion if there is an atlas $\widetilde{\tau}$ on X such that $f^{-1}(\widetilde{\tau})$ is an atlas on X' such that for $V \in \widetilde{\tau}$ there is $V' \in \mathrm{F}^{-1}(\widetilde{\tau})$ with a correspoinding map on coordinate rings $\mathscr{A}_V \to \mathscr{A}_{V'}$ which is a surjection.

However, one has to pay special attention to k-analytic spaces that are not good.

Question. Is this equivalent to requiring that for the maximal atlas $\widehat{\tau}$ on X, $f^{-1}(\widehat{\tau}) \subset \widehat{\tau}'$ is an atlas on X' and satisfies the corresponding property as abve?

Using this, we may define the notion of separatedness.

Definition 3.2. A morphism of k-analytic spaces $f : X' \to X$ is called separated, if the diagonal map $\Delta_f : X \to X \times_{X'} X$ is a closed immersion.

Note that a diagonal with closed image might not be a closed immersion. This is due to the fact, that for the glueing in the definition, we used atlases instead of topological open covers. This accounts for the difference between separatedness and the Hausdorff property for k-analytic spaces as the latter one is a purely topological property.

Example 3.3. There are compact Hausdorff k-analytic spaces that are not separated over $\mathscr{M}(k)$.

More explicitely, the distinction between the two conditions is due on the on hand to the difference between topological and k-analytic fibre products as already discussed before; on the other hand diagonal maps in k-analytic spaces may lack, because of the glueing proces, a kind of immersion property, that they always have in category of spaces, where the topology is used for the gluing.

On the other hand, purely for the underlying topological spaces, one can show (classically) that Hausdorff and separated are equivalent. That is, for a map of k-analytic spaces $f : X' \to X$, the induced map on the underlying topological spaces $|f| : |X'| \to |X|$ is separated (in other words $\Delta_{|f|}$ is a closed embedding) if and only if it is Hausdorff (i.e. preimages of Hausdorff subsets are Hausdorff). By contrast, if $f : X' \to X$ is separated, then $|f| : |X'| \to |X|$ is separated, so Hausdorff, but the converse is false in general. In particular, if X is separated over $\mathscr{M}(k)$ then X is Hausdorff.

4 Analytification functors

We want to describe the relations between k-analytic spaces, algebraic k-schemes and rigid-analytic spaces over k. We assume here (as we did for the most part in the rigid part) that k has non-trivial absolute value.

Recall that for any quasi-compact and quasi-separated rigid space, is can be given by gluing finitely many k-affinoids along quasi.compact admissible opens. By further choosing a finite, thus admissible cover of each quasi-compact overlap by affinoid opens, we have a characterisation of X in terms of finitely many k-affinoid spaces and affinoid subdomains.

This can be caried over to the category of k-analytic spaces using atlases instead of admissible covers, and for the above X a compact Hausdorff strictly k-analytic space X^{an} called the analytification of X. This construction is independent of the choice of covering of X (as one can on the k-analytic side pass to the maximal atlas) and it yields a functor

$$\left\{\begin{matrix}\text{quasi-compact and quasi-separated}\\ \text{rigid spaces}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{compact Hausdorff strictly}\\ k\text{-analytic spaces}\end{matrix}\right\}.$$

This functor adds points to a rigid space: the underlying set of X is the set of points $x \in X^{\mathrm{an}}$ that have coefficients in a finite extension field of k, $[\kappa(x) : k] < \infty$. An example shows that k-analytic spaces that arise in this way may fail to be good – the only ones considered in Berkovich's first paper [2] – which are therefore inadequat for a satisfactory general theory that.

Example 4.1. Assume that k has non-trivial absilute value. Consider the admissible open locus (in rigid analytic spaces)

$$X = \left\{(t,t') \in \mathbb{B}_k^2 \quad | \quad |t'| = 1\right\} \cup \left\{(t,t') \in \mathbb{B}_k^2 \quad | \quad |t| = 1\right\}$$

that is the union of two affinoid subdomains and the complement of the open unit polydisc. The associated k-analytic space X^{an} is a k-analytic domain in $\mathscr{M}(k\langle t,t'\rangle)$ that is not good. More explicitely, the point ξ representing the Gauß norm lies in X^{an} but does not admit a k-affinoid neighbourhood. This can be seen using results by Temkin [3].

We list some properties.

- one can show that the above functor is even an equivalence of categories
- and is compatible with fibre products and extension of the base field.
- Let $f : X' \to X$ be a morphism between quasi-compact and quasi-separated rigid spaces and $f^{\mathrm{an}} : X'^{\mathrm{an}} \to X^{\mathrm{an}}$ the induced morphism of k-analytic spaces. Then f is a closed immersion if and only if f^{an} is a closed immersion.
- Applying this to the diagonal map givs the same result for separated morphisms.
- It follows that if X is a quasi-compact and quasi-separated rigid space that is not separated then X^{an} is a compact ausdorff strictly k-analytic space that is not separated over $\mathscr{M}(k)$.

One can do the same procedure of analytification without quasi-compactness, but in that case, one has to impose a finiteness codition on both sides of the equivalence. We wont go into details here.

There is also a natural analytification functor from

$$\left\{\begin{matrix}\text{algebraic}\\ k\text{-schemes}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{strictly}\\ k\text{-analytic spaces}\end{matrix}\right\}$$

compatible with fibre products and extension of the base field as well. Moreover it is compatible with the analytification functor from algebraic k-schemes to rigid spaces over k and with the above introduced one from rigid spaces (with certaine conditions) to certaine strictly k-analtic spaces.

We said earlier, that the analytification functor adds points on the underlying topological spaces. Interestingly, if one considers a k-analytic space and removes a non-classical point, that is a point that doesn't appear in the uncerlying topological spaces of the associated rigid space, this can still have an impact on the associated rigid space, even though one couldn't se the point topologically in teh first place. Let us consider an example.

Example 4.2. Let k be as above a non-archimedean field with non-trivial absolute value, and $X = \mathscr{M}(k\langle T\rangle)$, with the open immersion $U = X - \{\xi\} \to X$ where ξ is the point corresponding to the Gauß norm. Both are paracompact Hausdorff strictly k-analytic spaces. What can we say about the associated rigid analytic spaces — considering the fact, that the Gauß point has infinite degree over k, so it doesn't appear in the rigid space, which implies, that at least the underlying topological spaces of the rigid spaces associated X_0 and U_0 to X and U coincide. However, it is another matter with the rigid spaces themselves. It is clear that $X_0 = \mathbb{B}^1_k$. On the other hand, one can show that U_0 is a disjoint union of twisted open unit discs labeled by the closed points of $\mathbb{A}^1_{\tilde{k}}$ (if k is already algebraically closed, then U_0 is a union of ordinary unit discs). In particular, the map $U_0 \to X_0$ is a bijective local isomorphism that is not an isomorphism. Indeed, it identifies U_0 with the disjoint unioin of the open residue discs for $\mathbb{B}^1_k$ (which are by the way a cover of $\mathbb{B}^1_k$ which is non-admissible).

Question. This question seems interesting to me, but I didn't have time to think about: Let X be a quasi-compact and quasi-separated rigid space, and X^{an} the associated Berkovich space via the analytification funtor. What conditions do we have to impose on X, for X^{an} to be good?

5 A Grothendieck topology on Berkovich spaces

As we have seen, and will later come to again, Berkovich analytic spaces have a very nice topology. Unfortunately, it is not compatible with the glueing process used to globalise. Of course one could use quasi-nets to define a Grothendieck topology on a k-analytic space much like we did in the case of rigid analytic spaces. Of course this is not an actual topology and loses many of the good properties that the natural topology of a k-analytic space has. The objects of the G-topology are the k-analytic subdomains $Y \subset X$ and a covering $\{Y_i\}$ of Y is a set-theoretic covering, which satisfies the local finiteness condition of quasi-nets (each $y \in Y$ has a neighourhood of finitely many Y_i's such that it's contained in the intersection $y \in \bigcap Y_i$. This corresponds to the local finiteness property of admissible covers in rigid geometry. We denote by X_G the space X considered with this Grothendieck topology. As we can conclude from what we said last time about affinoid subdomains, there is a unique way to define a structure sheaf on this space. Restricting this sheaf to open sets in the natural topology of X gives us a sheaf on X.

However, as we have seen, affinoids are generally not open in the natural topology. In particular, if a k-analytic space is not good, it is difficult to work with stalks, because there are points then that lack affinoid neighbouroods. This makes flatness as well to a difficult concept in this context. For example, one an construct examples of coordinate rings of curves, whose underlying topological space is in fact a point, and becomes only a curve after sufficient extension of the base field (e.g. $k\langle r^{-1}X, rX^{-1}\rangle$). Sometimes it is possible to first treat the good case in a proof, and then by gluing arguments "bootstrap" to the general case. One example for this would be a theory of coherent sheaves. Another one is the definition of finite morphisms.

Definition 5.1. A map $f : X' \to X$ of k-analytic spaces is finite if, for all $x \in X$ there exist k-afinoid subdomains $V_1, \dots V_n \subset X$ such that the union $\bigcup V_i$ is a neighbourhood of x in X, $x \in \bigcup V_i$, $x \in \bigcap V_i$, and the preimages $V'_j = f^{-1}(V_j) \subset X'$ are k-affinoid subdomains in X' such that the induced morphism $\mathscr{A}_{V_j} \to \mathscr{A}_{V'_j}$ of Banack algebras is finite and compatible with norms.

This is a local property in the sense that the requirements on the pair (V_i, V_i') holds for every pair $(V, f^{-1}(V))$.

6 When Berkovich spaces are useful

There are examples when passing to Berkovich theory makes proofs easier and more intuitive. WE only mention a couple of them.

Examples 6.1.

- Let $X = \mathrm{Sp}(A)$ and $Z = \mathrm{Sp}(A/I)$ for some ideal $I \subset A$ with a set of generators $\{f_1, \dots, f_n\}$. Let U be an admissible open that contains Z. For $\epsilon > 0$ we can consider the tube $\{|f_1| \leqslant \epsilon, \dots, |f_n| \leqslant \epsilon\}$ around Z. The claim is that there exist ϵ such that this tube is contained in U. This can be proved within the framework of rigid geometry, but is much more accessible if translated to k-analytic geometry. This fact also plays a role in the definition of rigid cohomology.
- The theory of properness.
- The theory of étale morphisms.

References

[1] Brian Conrad. Several approaches to non-archimedean geometry. In *p-adic Geometry*, volume 78 of *AMS University Lecture Series*. Amer. Math. Soc., Providence, RI 41, 2008. Lectures from the 2007 Arizona Winter School.

[2] Vladimir G.Berkovich. *Spectral theory and analytic geometry over non-archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1990.

[3] M. Temkin. On local properness of non-archimedean analytic spaces ii. *Israeli Journal of Math*, 140:1–27, 2004.