Introduction

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We may ask the question, what actually rigid geometry should be – and what was it meant to be? Essentially, one can think of an analogue of complex analysis over non-archimedean valued fields. However, over the years it has acquired a much richer structure, so that it goes beyond of just being a copy of complex analysis. This opened the door for a variety of applications, and it became an important and independent discipline in arithmetic and algebraic geometry.

Let me give you some background. Much of this talk is based on the remarkable introduction in the book of Fujiwara and Kato [2].

1 History

When Hensel introduced the p-adic numbers at the end of the 19^{th} century, the idea came up of constructing p-adic analogues of concepts known and studied over real and complex numbers, such as the concept of analytic functions (which by the work of Cartan, Serre, Oka should be seen as part complex analytic geometry).

The first attempts ran into problems. The "naive" approach, as adopted for example by Serre in some lectures at Harvard, was to copy the ideas over $\mathbb{C}$ almost verbatim and adapt what was needed to prove similar theorems. For example the "complex plane" over $\mathbb{C}_p$ would be the naive point set, namely $\mathbb{C}_p$ itself, together with the topology arising from the p-adic valuation. Then one could try to construct locally ringed spaces $(X, \mathscr{O}_X)$ where $\mathscr{O}_X$ should correspond to holomorphic functions. This means, one defines for each open set $U \subset X$

$$\mathscr{O}_X(U) = \{ \text{ functions that admit a convergent power series expansion at every point } \}$$

And here arises the problem: the induced topology by the valuation ν_p is totally disconnected. In essence, this means that there are "too many" opens, and if one wants to glue/patch functions, one gets a very unstable theory in the sense, that one has no good control over the global behaviour of analytic functions. (Compare this to the usual complex analysis – which is rather rigid!)

Example 1.0.1. The analogue of the Riemann sphere, the p-adic Riemann sphere

$$X = \mathbb{C}_p \cup \{\infty\}$$

and the set of global section $\mathscr{O}_X(X)$ should only be constant functions. But this is not the case!

This is called the Problem of Analytic Continuation, or the Globalisation Problem. This seems to be a topological problem, which however is very deeply (and maybe surprisingly) linked to the notion of point.

2 Tate's approach

Tate gave fundamental solution in the early 60's to the globalisation problem, when he introduced Tate curves – a non-archimedean analogue of one dimensional complex tori [4].His solution basically was to define a reasonable and sufficiently large class of analytic functions and a correct notion of coverings. Let's take a closer look at this.

Let E be an elliptic curve over $\mathbb{C}$. By Weierstraß this can be represented canonical as a torus $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\,\alpha + \mathbb{Z}\,\beta$ is a lattice in the complex plane. This parametrisation gives much more transparence to the theory. After rescaling, we can assume that Λ is of the form $\mathbb{Z}+\mathbb{Z}\tau$ with $\tau \in \mathfrak{H}$ the upper half-plane. As a consequent, functions that generate the field of meromorphic functions are doubly periodic and in particular invariant under translation by 1. Thus one can write them as Fourier series, and the exponential map provides a map

$$\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \xrightarrow{\exp} \mathbb{C}^*/q^{\mathbb{Z}},$$

with $q = e^{2\pi i \tau}$ given by τ.

Now we want to transfer this to the p-adic setting. Let $K/\mathbb{Q}_p$ be finite, and $|\cdot|$ be the normalised absolute value induced by the p-adic valuation ν_p. Analogous to the above, we want to take something like

$$\mathbb{G}_m/q^{\mathbb{Z}}$$

over K. Schemes don't work. But we can imitate complex analysis, and on the level of points indeed everything works fine. Tate proved the following:

Theorem 2.0.2 (Tate)**.** *Let $K/\mathbb{Q}_p$ be a finite extension, E an elliptic curve over K with j-invariant $|j(E)| > 1$. Then there is a unique $q \in K^*$, with $|q| < 1$ such that $E \cong E_q$ over the algebraic closure $\overline{K}$, where E_q is given in coordinates as power series*

$$\begin{aligned} \Phi : \overline{K}^* &\to E_q(\overline{K}) \\ u &\mapsto \begin{cases} (X(u,q), Y(u,q)) & \text{if } u \notin q^{\mathbb{Z}} \\ 0 & \text{if } u \in q^{\mathbb{Z}} \end{cases} \end{aligned}$$

On the level of points all is good. But interestingly, this approach also gives a clue how to solve the globalisation problem. A first glance on the object that we defined reveals, that one can't use usual scheme theory but has to use power series instead. And this was Tate's quite ingenious idea: use concepts from algebraic geometry in the Grothendieck sense, but on power series rings.

He replaces finite K-algebras A by a certain type of power series algebras $\mathscr{A}$, so called affinoid algebras. Similar to $\operatorname{Spec} A$ the dual object, denoted by $\operatorname{Sp}\mathscr{A}$ gives the associated space. The set of points – seen as a sort of visualisation – is given by the maximal ideals of $\mathscr{A}$. However, the notion of topology has to be weakened compared to Zariski topology on $\operatorname{Spec} A$ which is an honest topology: Tate introduces the notion of admissible covers, thereby building a Grothendieck topology. This rigidifies the point-set topology, coming from the p-adic valuation enough to make geometry possible and solve the globalisation problem. At the same time, this proved to be useful to define coherent sheaves and cohomologies on rigid spaces.

3 Further developements

But there were still some deficiencies. For one, functoriality of points does not hold. For an extension K'/K of complete non-archimedean valued fields, one should have for a rigid space X/K a map from the base change $X_{K'}$ to X. But this is not in general true.

Closely related to this, the Gelfand-Mazur theorem does not hold. Over $\mathbb{C}$ it tells us, that there are no Banach field extensions of $\mathbb{C}$ other than itself. In contrast in the non-archimedean case there are many other than finite extensions of K. Thus there should be much more valued points of affinoid algebras not factoring through the residue field of a maximal ideal.

Both of these statements suggest that there should be more than the naive points. We have to change the notion of points, and the definition of spectra. One idea is to realise that points correspond to valuations. This is taken up in what is now known as Berkovich theory [1]. The associated class of spectra are called Gromov–Berkovich spectra, and are the smallest spectra to solve the functoriality problem. It provides a relatively flexible notion in that they deal with a wide range of Banach algebras. Additionally, the resulting natural topology is Hausdorff, which is not the case for the admissible topology. However, it also behaves counter intuitive, is in fact weaker than the admissible topology. You might have heard the slogan that open sets in Berkovich spaces are closed in affinoid spaces. It does not solve the globalisation problem satisfactorily.

Another approach is the Zariski–Stone approach, where one considers higher valuations, thus creating more points. The amazing thing is, that the induced point-set topology coincides with the admissible topology after restriction. This approach is for example taken by Huber in his adic space theory.

Very closely related to this is Raynaud's approach via formal geometry [3]. He realised in fact that formal geometry can serve as a model for rigid geometry. In other words, Tate's geometry arises from formal geometry as a quotient (by admissible modifications). This was taken up by Fujiwara and Kato and developed further [2].

References

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[2] Kazuhiro Fujiwara and Fumiharo Kato. *Foundations of Rigid Geometry.* Monographs in Mathematics. European Mathematical Society Publishing House, to be published.

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[4] John Tate. Rigid analytic spaces. *Inventiones Mathematicae*, 12(4):257–289, 1971.