Coherent sheaves on rigid-analytic spaces

Graduate Student Seminar in Arithmetic Geometry
Universität Regensburg

Winter term 2014/15

In ordinary scheme theory coherent sheaves play an important role. Recall that for a scheme $(X, \mathscr{O}_X)$ a sheaf $\mathscr{F}$ is called quasi-coherent if X can be covered by open affines $\operatorname{Spec} A_i$ such that there is an A_i-module M_i with $\mathscr{F}|_{\operatorname{Spec} A_i} \cong \widetilde{M}_i$, the by M_i induced sheaf. $\mathscr{F}$ is called coherent, when we can chose a finite module. We will imitate this in the case of rigid analytic spaces. We follow closely [2, Section 3.1].

1 Coherent sheaves on affinoid and rigid spaces

To get started we need to construct an analogue to the sheafification of a module.

Definition 1.1. Let $X = \operatorname{Sp} A$ be affinoid over k, and M a finite A-module. The assignment for each affinoid subdomain

$$U \mapsto A_U \otimes_A M$$

defines uniquely an $\mathscr{O}_X$-module $\widetilde{M}$.

Kiehl proves the following.

Theorem 1.2. *With notations as above, we have*

$$M \cong \widetilde{M}(X)$$

and the natural map (global section functor)

$$\operatorname{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F}) \to \operatorname{Hom}_A(M, \mathscr{F}(X))$$

is bijective for any $\mathscr{O}_X$-module $\mathscr{F}$.

This can readily be globalised. We define coherent sheave by using admissible affinoid covers instead of affine covers. Kiehl shows that this makes sense.

Theorem 1.3. *Let X be a rigid space, and $\{U_i\}$ an admissible affinoid cover. For an $\mathscr{O}_X$-module $\mathscr{F}$ the following are equivalent.*

1. *For every admissible affinoid open $V \subset X$, there is a finite $\mathscr{O}_X(V)$-module M_V such that*

$$\mathscr{F}|_V \cong \widetilde{M}_V.$$

2. *For every i there is a finite $\mathscr{O}_X(U_i)$-module M_i such that*

$$\mathscr{F}|_{U_i} \cong \widetilde{M}_i.$$

This is shown by using a relatively simple standard cover. And then use Tate acyclicity.

Definition 1.4. We call such a module coherent. We say, that $\mathscr{F}$ is locally free of rank n, if each M_i is free of rank n. We say $\mathscr{F}$ is invertible, if it is free of rank 1.

We know that in the case of schemes quasi-coherence carries over to kernel, cokernel, image of a map, extensions and tensor products. If we assume additionally X to be Noetherian, this is also true for coherent sheaves. And in fact this is equally true in the rigid analytic setting (in the coherent case, without assuming Noetherian). To show this, one works locally (the previous result shows, that it's a local question). Then one uses the equivalence theorem of Kiehl. One shows that the functor $M \mapsto \widetilde{M}$ is exact and fully faithful. From this one gets directly the part about kernels, cokernels, images, tensor products. For extensions, one uses additionally the five lemma.

2 Quasi-coherent sheaves

Most of this so far is almost a copy of usual scheme theory. However, problems arise, when one wants to define quasi-coherent modules, i.e. we want to drop the finiteness condition. This can be interpreted as taking direct limits of coherent sheaves. This definition has some nice properties: preserved under kernel, cokernel, image, extensions tensor product, direct limits. However, it does not globalise nicely. I.e. the equivalent of the previous theorem is not true: on an arbitrary admissible affinoid, it might not be the direct limit of coherent sheaves.

Example 2.1. Gabber's example: find a sheaf of modules on the unit disk $\mathbb{B}_1 = M(T_1) = M\langle t\rangle$ such that $\mathscr{F}$ is locally the direct limit of coherent sheaves, but not on the whole unit disk. Namely, coherent sheaves have vanishing higher cohomology, and since taking cohomology commutes with taking direct limits, this should be true for the limit of coherent sheaves. However, it turns out that Gabber's example has non-vanishing H^1. See for this [1, Example 2.1.6]

Choose two rational points $x' \neq x'' \in \mathbb{B}_1(k)$ of the unit disk over k, and let $U' = \mathbb{B}_1 \setminus \{x'\}$ and $U'' = \mathbb{B}_1 \setminus \{x''\}$ the associated open sets, and $U = U' \cap U''$. On each of the open sets U' and U'' define free sheaves of countbly infinite rank

$$\mathscr{F}' = \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_{U'}\, e'_n \quad \textit{and} \quad \mathscr{F}'' = \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_{U''}\, e''_n$$

One can glue them on the intersection U to obtain a sheaf $\mathscr{F}$ on $\mathbb{B}_1$ in a way such that it has no nonzero global sections (think of glueing the leaves together with enough knots).

Then by definition $\mathscr{F}$ is quasi-cherent, because $\mathscr{F}\big|_{U'}$ and $\mathscr{F}\big|_{U''}$ are direct limits of coherent sheaves. To compute cohomology of $\mathscr{F}$ we use the following trick: multiplication with the standard coordinate t from above ($T_1 = k\langle t\rangle$) gives a short exact sequence

$$0 \to \mathscr{F} \xrightarrow{t} \mathscr{F} \to \mathscr{F}/t\mathscr{F} \to 0$$

and the associated long exact sequence of cohomology provides an injection

$$\mathrm{H}^0(X, \mathscr{F}/t\mathscr{F}) \hookrightarrow \mathrm{H}^1(X, \mathscr{F})$$

because $\mathrm{H}^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F}) = 0$. However, $\mathscr{F}/t\mathscr{F}$ is a nonzero skyscraper sheaf suported at the origin, and hence $\mathrm{H}^0(X, \mathscr{F}/t\mathscr{F}) \neq 0$. Thus $\mathrm{H}^1(X, \mathscr{F}) \neq 0$ and cannot be the direct limit of coherent sheaves.

Remark 2.2. We will talk about cohomology later. For the example it was important, that taking sheaf cohomology commutes with taking direct limits of sheaves on a rigid analytic space. This has to be shown in the rigid analytic setting. And one does with a Čech spectral sequence argument.

Apparently, no "reasonable" definition of quasi-coherence has been found in rigid analytic geometry.

3 Stalks of coherent sheaves

Recall, that we said, in general stalks of sheaves don't behave nicely on rigid analytic spaces, more precisely, one can find examples, of a section s that is non-trivial, but is trivial on stalks. This was because in the definition of a stalk we included potentially coverings that are not admissible. However, in the case of coherent sheaves, this pathology does not occur.

Let $X = \operatorname{Sp} A$ be affinoid. Then using definitions, it is easy to see, that for a finite A-module M

$$\widetilde{M}_x \cong \mathscr{O}_{X,x} \otimes_A M$$

for all points $x \in X$. By definition

$$\widetilde{M}_x = \varinjlim \widetilde{M}(U)$$

where the limit is taken over admissible opens or affinoid subdomains containing x exchangably. Taking the letter class of subdomains, means, we can write

$$\widetilde{M}_x = \varinjlim A_U \otimes_A M$$

but $\varinjlim A_U = \mathscr{O}_{X,x}$, and the claim follows.

Recall that the ring $\mathscr{O}_{X,x}$ is local noetherian and faithfully flat over A_x and has the same completion. Now $m \in M$ vanishes in $\widetilde{M}_x$ for all x, if and only if, it vanishes in

$$\widehat{\mathscr{O}}_{X,x} \otimes \widetilde{M}_x = \widehat{A_x} \otimes \widetilde{M}_x$$

if and only of it vanishes in $M \otimes_A A_x$. As $A = \bigcap_{x \in X} A_x$, it follows that this is the case if and inly if $m = 0$ already in

$$M = M \otimes_A A.$$

More globally, recall that the stalk of a sheaf can be defined in the same way

$$\mathscr{F}_x = \varinjlim \mathscr{F}(U)$$

taking the limit over affinoid neighbourhoods of x. In the case of a coherent sheaf, this means

$$\mathscr{F}_x = \mathscr{O}_{X,x} \otimes_{A_V} M_V$$

for any admissible affinoid open containing x. By aboves argumentation, a global section s of $\mathscr{F}$ is zero for one stalk, if and only if it is zero on $\operatorname{Sp} A_V$ for each admissible affinoid open V containing x. It follows that s is zero for every stalk, if and only if it vanishes for every admissible affinoid open of X. By the sheaf property, it follows that this happens only if s is zero globally.

This property fails for general abelian sheaves, as we have already seen in Gabbers example explicitely.

4 Kähler differentials

Recall the usual definition of Kähler differentials. Let $A \to B$ be a map of k-affinoid algebras. Let I be the kernel of the natural map

$$B \otimes_A \mathrm{B} \to B$$

induced by multiplication. Then the B-module I/I^2 gives rise to a complex together with a boundary map. However, this is usually huge, and in particular not finite over B, since the usual tensor product $B \otimes_A B$ might not be Noetherian. To remedy this, replace $\otimes$ by the completed tensor product $\widehat{\otimes}$ as described earlier. With this definition I/I^2 is a finite B-module that gives rise to a coherent sheaf $\Omega^1_{B/A}$ together with a $\mathscr{O}_A$-linear derivation $\mathscr{O}_B \to \Omega^1_{B/A}$. One easily extends this to a complex.

Question: is this reminiscent of the fact, that in MOnsky–Washnitzer cohomology, one considers contiuous differentials?

5 Separated and quasi-separated rigid spaces

We make definitions inspired by scheme theory concerning morphisms between rigid spaces in order to define separated and quasi-separated spaces.

Definition 5.1. A rigid-analytic space X is said to be quasi-compact if it has an admissible covering consisting of finitely many affinoid opens. A morphsim $f : X' \to X$ of rigid-analytic spaces is said to be quasi-compact, if there is an admissible cover of X by affinoid opens $\{U_i\}$, such that such that each preimage $f^{-1}(U_i)$ is quasi-compact.

Moreover, f is called a closed immersion, if there is an admissible affinoid covering $\{U_i\}$ of X such that each preimage $U_i' = f^{-1}(U_i)$ is affinoid, and the ristricted map $U_i' \to U_i$ of affinoids corresponds to a surjection on coordinate rings.

This is a local property in the sense that for every admissible affinoid open, the preimage U' is affinoid, and the induced map $U' \to U$ corresponds to a surjection on the coordinate rings. What is more, the induced map on structure sheaves

$$f_*(\mathscr{O}_X) \to \mathscr{O}_{X'}$$

is surjective, with coherent ideal sheave $\mathscr{I}$. And conversely, every coherent ideal sheaf arises in his way. (Let $\mathscr{I} \subset \mathscr{O}_X$ be an ideal sheaf, for an admissible affinoid open, $\operatorname{Sp}(A) \subset X$ let $I = \mathscr{I}(\operatorname{Sp}(A)) \subset \mathscr{O}_X(\operatorname{Sp}(A)) = A$, then glue.)

Now we are in a position to make our definitions.

Definition 5.2. A map $f : Y \to X$ of rigid analytic spaces, is separated if the diagonal map

$$\Delta_f : X \to X \times_Y X$$

is a closaed immersion. In case $Y = \mathrm{Sp}(k)$ is a point, we say, that X is separated.

Definition 5.3. I Δ_f is quasi-compact, we say that f is separated.

Definition 5.4. We say a map of rigid spaces $f : X' \to$ is finite, if there is an admissible affinoid covering $\{U_i\}$ of X such that the preimages $U'_i = f^{-1}(U_i)$ are affinoid, and the induced map of coordinate rings

$$\mathscr{O}_X(U_i) = A_{U_i} \to \mathscr{O}_{X'}(U'_i) = A_{U'_i}$$

is module-finite.

Lemma 5.5. *The property of finiteness is local in the sense that for any affinoid open $U \subset X$ the preimage $f^{-1}(U)$ is affinoid with finite coordinate ring over that of U.*

Proof. Use the universal property to show that preimages of affinoids are affoÃnoid. Finiteness is clear then. □

It is crucial to assume finiteness as there is no analogue of affine morphisms in scheme theory. In scheme theory, one can use Serre's cohomological criterion for affineness: a scheme is affine, if and only for all quasi-coherent sheaves $\mathscr{F}$ higher cohomology groups $\mathrm{H}^{i>0}(X, \mathscr{F}) = 0$ vanish. In rigid geometry by contrast one can find examples of quasi-compact separated non-affinoid spaces whose coherent sheaves have vanishing higher cohomology groups. One can find examples with even more pathologies.

Example 5.6. See Liu's article [3].

Lemma 5.7. *Let $f : X \to Y$ be a map of rigid spaces and Y separated. Then for any admissible affinoid opens $U \subset Y$ and $V \subset X$ the intersection $V \cap f^{-1}(U)$ is again affinoid.*

Proof. This can be carried over from scheme theory. It is clear that the induced map

$$(\mathrm{id}, f) : X \to X \times Y$$

is a closed immersion. Then for $U \subset Y$ and $V \subset X$ affinoid opens, the fibre product $V \times U \subset X \times Y$ is affinoid as well, as we have already discussed. The preimage under the map (id, f) is exactly the intersection we are looking for, which is again affinoid, since (id, f) is a closed immersion. □

Corollary 5.8. *The intersection of finitely many admissible affinoid opens in a separated rigid space is again affinoid.*

Proof. Take f in the previous proof to be the identity, and do this successively finitely many times. □

One can show a different characterisation of quasi-separatedness.

Lemma 5.9. *Let X be a rigid space. X is quasi-separated if and only if it has an admissible covering by admissible affinoid opens such that the intersections are quasi-compact.*

Corollary 5.10. *Quasi-separatedness is a local property in the obvious way.*

Lemma 5.11. *Let X be a quasi-separated rigid space. For any finite collection $\{U_i\}$ of quasi-compact admissible opens in X the union $U = \bigcup U_i$ is an admissible open with covering the U_i's.*

Example 5.12. An example of a quasi-compact rigid space which is not quasi-separated?

6 Extension of the base field

Let K/k be an analytic extension. On the category of quasi-separated rigid spaces over k, one can define a base change functor to rigid spaces over K

$$X \mapsto X_K,$$

compatible with fibre products. this is very useful for example in the case $K = \widehat{\overline{k}}$ (e.g. $k = \mathbb{Q}_p$ and $K = \mathbb{C}_p$).

Construction in the affinoid case:

$$A \mapsto K\widehat{\otimes}_k A$$

using the completed tensor product explained some sessions ago.

In the separated case, one can glue this construction as intersections of affinoids are affinoid. In the quasi-separated case, this may fail, i.e. the intersection of affinoids may not be affinoid, but is at least quasi-compact and separated. Now one uses the argument of the separated case on this intersection, and glues again.

However, one has to bear in mind, that this is just a construction and generally not functorial nor in any way universal (if the degree of K/k is infinite). This comes from the fact, that the maximal spectrum, contrary to the usual ring spectrum, is not functorial with respect to base change. A remedy to this is found by Berkovich and Huber respectively in their approaches to rigid geometry, as they allow many more points in the underlying spaces, which makes it possible to see the base change functor actually as finer product (which is functorial!).

If it was functorial, there would be a map $X_K \to X$, but the transcendental points have nowhere to go!

Chnage of the base field exhibits further pathologies, since elements in $k\langle \underline{x} \rangle$ can have infinitely many non-zero coefficients (contrary to polynomials).

Example 6.1. Consider a non.archimedean field of characteristic $p > 0$ such that $[k : k^p] = \infty$. For example, take a field F of characteristic p such that $[F : F^p] = \infty$ (I think $F = \mathbb{F}_p(x_{\mathbb{N}})$ should work). Then $k = F((y))$ where y is a variable, together with the y-adic valuation is such a field. Now take the sequence, $a_n = Y^n$. It is clear, that in the y-adic topology, this tends to 0 and furthermore, $|a_n| \leqslant 1$. Then the p^{th} roots $a_n^{\frac{1}{p}}$ generate an infinite degree extension K/k. Now base extension of certain Tate algebras/ affinoid algebras can change reducedness and normality.

Example 6.2. Base extension can also have some effects on admissible opens, for example for the inclusion of an admissible open.

References

[1] Brian Conrad. Relative ampleness in rigid analytic geometry. *Annales de l?institut Fourier*, 56(4):1049–1126, 2006.

[2] Brian Conrad. Several approaches to non-archimedean geometry. In *p-adic Geometry*, volume 78 of *AMS University Lecture Series*. Amer. Math. Soc., Providence, RI 41, 2008. Lectures from the 2007 Arizona Winter School.

[3] Qing Liu. Un contre-exemple au «critére cohomologique d'affinoïdicité ». *C. R. Acad. Sci. Paris*, 307:83–86, 1988.