The motivic complex of Suslin-Voevodsky

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The existence of motivic cohomology fro smooth varieties was conjectured by Beilinson and Lichtenbaum. This was stimulated by observations of Suslin and Nesterenko-Suslin concerning Milnor and Quillen K-groups of fields and later local rings. Essentially, motivic cohomology with coefficients in an abelian group A is a (bigraded) family of contravariant functors

$$\mathrm{H}^{pq}(-, A) : \mathscr{S}m\,/k \to \mathscr{A}b$$

from smooth schemes over a given field k to abelian groups, satisfaing two sets of (conjectural) properties. One set concerns motivic cohomology itself, for example

- homotopy invariance
- Mayer-Vietoris sequence
- Gysin sequence
- Beilinson-Soulé vanishing conjecture ($\mathrm{H}^{pq} = 0$ for $p < 0$)
- Beilinson-Lichtenbaum conjecture (étale descent: $\mathrm{H}^{pq}_{\mathrm{Zar}} \to \mathrm{H}^{pq}_{\text{ét}}$ is an isomorphism)

The other set of properties relates motivic cohomology to other invariants of varieties. Considering the introductory remarks it is not surprising that one of Beilinson and Lichtenbaum's conjectures was that for an essentially smooth local ring A over a field, there should be an isomorphism

$$K_n^M(A) \xrightarrow{\sim} \mathrm{H}^{n,n}(A, \mathbb{Z}).$$

This was established by Kerz for regular local rings with "big enough" residue fields. Some elementary comparison results that have been established are

- $\mathrm{H}^{p,q}(X, A) = 0$ for $q < 0$ and if X is connected

  $$\mathrm{H}^{p,0}(X, A) = \begin{cases} A & \text{for } p = 0 \\ 0 & \text{for } p \neq 0 \end{cases}$$

- One has

  $$\mathrm{H}^{p,1}(X, \mathbb{Z}) = \begin{cases} \mathscr{O}^*(X) & \text{for } p = 1 \\ \mathrm{Pic}(X) & \text{for } p = 2 \\ 0 & \text{otherwise} \end{cases}$$

- For a field k

  $$\mathrm{H}^{p,p}(k, A) = K_p^M(k) \otimes A$$

  which is a special case of the above mentioned conjecture.

- For a strictly Hensel local scheme S/k and n prime to $\operatorname{char} k$

$$\mathrm{H}^{p,q}(S, \mathbb{Z}/n) = \begin{cases} \mu_n^{\otimes q}(S) & \text{for } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

- for Bloch's higher Chow groups $\mathrm{H}^{p,q}(X, A) = \mathrm{CH}^q(X, 2q - p; A)$ and in particular

$$\mathrm{H}^{2q,q}(X, A) = \mathrm{CH}^q(X) \otimes A$$

Often Bloch's construction is used as definition of motivic cohomology. In general, Bloch's definition is better suited for studying mod-p phenomena in positive characteristic p. But it seems that Suslin-Voevodsky's definition is more apt to prove certain statements like the Beilinson-Lichtenbaum conjecture.

1 Finite correspondences

1.1 Definition of the category

The functor we want to define has as its domain smooth separated schemes (of finite type) over a perfect field k. However, the category $\mathscr{S}m/k$ with its usual morphisms is too rigid for some topological/homological phenomena such as homotopy equivalence. Instead, we consider a differnet kind of morphisms in this category – finite correspondences – and call the resulting category $\mathscr{C}or_k$, which contains $\mathscr{S}m/k$. All schemes considered are separated.

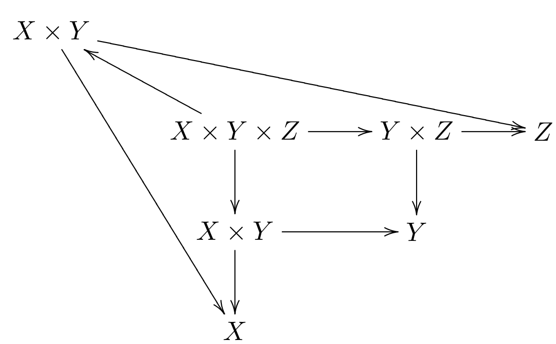
Definition 1.1.1. Let X/k be smooth and connected and Y/k any scheme. An elementary correspondence from X to Y is an irreducible closed subscheme $W \subset X \times Y$, such that the associated integral subscheme is finite and surjective over X. If X is not connected, we mean by this an elementary correspondence from a connected component of X.

We denote by $\mathscr{C}or(X, Y)$ the free abelian group generated by the elementary correspondences from X to Y. Its elements are called finite correspondences.

The definition implies that for a decomposition $X = \coprod X_i$ into connected components $\mathscr{C}or(X, Y) = \oplus \mathscr{C}or(X_i, Y)$.

1.2 Compositions

For compositions of finite correspondences it is enough to define them for elementary correspondences. Let $V \in \mathscr{C}or(X, Y)$ and $W \in \mathscr{C}or(Y, Z)$ elementary correspondences. We form the intersection product $[T] = (V \times Z) \cdot (X \times W)$ in $X \times Y \times Z$. The composition $W \circ V$ is then defined to be the push forward of $[T]$ along the projection $X \times Y \times Z \to X \times Z$. We have the following diagram



This is possible because $[T]$ is finite over $X \times Y$, so that the push-forward is defined and it is a standard argument that this defines a finite correspondence. The composition is associative and bilinear with identity $id_X = \Delta_X$ the correspondecne associated to the diagonal.

This defines the category $\mathscr{C}or_k$ and from the above remarks it is clear that this is an additive category with as zero object and disjoint union as coproduct.

1.3 Relation to the category $\mathscr{S}m/k$

We can see $\mathscr{S}m/k$ as a subcategory of $\mathscr{C}or_k$ by sending a morphism $f : x \to Y$ to its graph Γ_f. The identity element of $\mathscr{C}or(X, X)$ for the composition is the graph Γ_1 associated to the identity of X, which is the support of the diagonal $\Delta(X) \subset X \times X$. If in addition Y is smooth and X is connected, and $f : X \to Y$ is finite and surjective, the transpose of Γ_f is a correspondence from Y to X. The map

$$(f : X \to Y) \mapsto \Gamma_f$$

that sends a morphism of schemes to its graph is in fact a covariant faithful functor

$$\mathscr{S}m/k \to \mathscr{C}or_k .$$

We have already seen, that the graph Γ_1 associated to the identiy id : $X \times X$ is the identity for $\mathscr{C}or(X, X)$ and it is a standard computation that for the composition of two morphisms

$$\Gamma_g \circ \Gamma_f = \Gamma_{g\circ f}$$

1.4 Tensor product

A “tensor product” in $\mathscr{C}or_k$ can be defined in the following way.

Definition 1.4.1. Let $X, Y \in \mathscr{C}or_k$. Then we set

$$X \otimes Y = X \times Y.$$

If $V \in \mathscr{C}or(X, X')$ and $W \in \mathscr{C}or(Y, Y')$ then the cycle associated to the subscheme $V \times W$ defines a finites correspondence from $X \otimes Y$ to $X' \otimes Y'$.

This makes $\mathscr{C}or_k$ into a symmetric monoidal category.

Examples 1.4.2.

1. The set $\mathscr{C}or(k, X)$ is the set of zero-cycles of X. If W is a finite correspondence from $\mathbb{A}^1$ to x and $s, t : \operatorname{Spec} k \to \mathbb{A}^1$ are k-points, one can show that $W \circ \Gamma_s$ and $W \circ \Gamma_t$ are rationally equivalent.
2. Let $x \in X$ be a closed point conseidered as a correspondence from k to X. Then the composition

$$\operatorname{Spec} k \to X \to \operatorname{Spec} k$$

is multiplication by the degree $[k(x) : k]$.

2 Presheaves with transfers

2.1 Definition

In order to define motivic cohomology we need the notion of presheavs with transfer.

Definition 2.1.1. A presheaf with transfers is a contravariant additive functor

$$\mathscr{F} : \mathscr{C}or_k \to \mathscr{A}b$$

We denote this category by $\mathbf{PST}(k)$.

By additivity, there is a pairing $\mathscr{C}or(X, Y) \otimes \mathscr{F}(Y) \to \mathscr{F}(X)$. Restricting to the subcategory $\mathscr{S}m/k$ a presheaf with transfers may be seen as presheaf of abelian groups with an additional transfer maps $\mathscr{F}(X) \to \mathscr{F}(Y)$ for the finite correspondences from X to Y.

Examples 2.1.2. 1. A constant presheaf A on $\mathscr{S}m\,/k$ can be regarded as presheaf with transfers, where the associated transfer for a correspondence W is given by multiplication with $\deg(W)$ over X.

2. The sheaf $\mathscr{O}^*$ has a transfer induced by the usual norm map of fields.
3. Similarly for the sheaf $\mathscr{O}$ a transfer is induced by the usual trace map on function fields.
4. This can be extended to the Milnor K-sheaf as was shown by Kerz.
5. The Quillen K-sheaf K_0 does **not** admit a transfer.
6. The classical Chow groups are presheaves with transfers, where the transfer for a finite correspondence is given vie flat pull-back and intersection product.

In all these cases one has to verify that the transfer is compatible with composition in $\mathscr{C}or_k$.

Fact. If $\mathscr{F}$ is a presheaf with transfers, the associated Nisnevich/étale sheaf is naturally equipped with transfers as well. The functor that maps a presheaf to its Nisnevich sheaf

$$\alpha^{tr}_{Nis} : \mathbf{PST}(k) \to \mathbf{ST}(k)$$

is left adjoint to the inclusion $\mathbf{ST}(k) \subset \mathbf{PST}(k)$ and commutes with the functor that forgets the transfer. This is a priori wrong for Zariski topology.

Representable functors are another important category of presheaves with functors. By the Yoneda lemma, representable functors provide embeddings of $\mathscr{S}m\,/k$ and $\mathscr{C}or_k$ into the abelian category $\mathbf{PST}(k)$.

Definition 2.1.3. Let $\mathbb{Z}_{tr}(X) = \mathscr{C}or(-, X)$

If a morphism $Y \to X$ in $\mathscr{S}m\,/k$ is a Nisnevich cover, the induced morphism of sheaves

$$\mathbb{Z}_{tr}(Y) \to \mathbb{Z}_{tr}(X)$$

is an epimorphism. (And we even have an exact sequence

$$\mathbb{Z}_{tr}(Y \times X) \to \mathbb{Z}_{tr}(Y) \to \mathbb{Z}_{tr}(X) \to 0$$

for Nisnevich topology)

Definition 2.1.4. For a pointed scheme (X, x) we define $\mathbb{Z}_{tr}(X, x)$ to be the cokernel of the map $x_* : \mathbb{Z} = \mathbb{Z}_{tr}(x) \to \mathbb{Z}_{tr}(X)$ associated to the point $x : \operatorname{Spec} k \to X$. Since x_* splits the structure map, we have a natural splitting $\mathbb{Z}_{tr}(X) = \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x)$.

For our purposes the pointed scheme $\mathbb{G}_m = (\mathbb{A}^1 - 0, 1)$ will be important. We can also take products in the following way.

Definition 2.1.5. Let $(X_i, x_i)_i$ be a family of pointed schemes. We define $\mathbb{Z}_{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ to be

$$\operatorname{Coker}\left(\oplus \mathbb{Z}_{tr}(X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n) \xrightarrow{\mathrm{id} \times \cdots x_i \cdots \times \mathrm{id}} \mathbb{Z}_{tr}(X_1 \times \cdots \times X_n)\right)$$

and $\mathbb{Z}_{tr}((X, x)^{\wedge 0}) = \mathbb{Z}$.

2.2 Simplicial objects

Definition 2.2.1. Let Δ be the category whose objects are the natural numbers $n \in \mathbb{N}_0$. We denote them by $[n] = \{0, \ldots, n\}$. And order preserving functins as morphisms. For a category $\mathscr{C}$ a simplicial object is a covariant functor

$$\Delta \to \mathscr{C}$$

i.e. a presheaf of Δ with values in $\mathscr{C}$. Similar for cosimplicial objects.

Definition 2.2.2. Let

$$\Delta_k^n = \operatorname{Spec}(k[x_0, \ldots, x_n]/(\sum x_i - 1).$$

This defines a cosimplicial scheme $\Delta^\bullet$ over k where the face maps

$$\partial_j : \Delta^n \to \Delta^{n+1}$$

are given by the equation $x_j = 0$.

For simplicial and cosimplicial objects it is standard to define the associated complex. For a presheaf $\mathscr{F}$ of abelian groups on $\mathscr{S}m/k$, $\mathscr{F}(\Delta^\bullet)$ and $\mathscr{F}(U \times \Delta^\bullet)$ are simplicial abelian groups.

Definition 2.2.3. We write $C_\bullet \mathscr{F}$ for the simplicial presheaf $U \mapsto \mathscr{F}(U \times \Delta^\bullet)$ and $C_* \mathscr{F}$ for the associated complex.

They are both exact functors.

2.3 Homotopy invariance

Definition 2.3.1. A presheaf is homotopy invariant, if for every X the map

$$\mathscr{F}(X) \to \mathscr{F}(X \times \mathbb{A}^1)$$

is an isomorphism.

Let $i_\alpha : X \hookrightarrow X \times \mathbb{A}^1$ be the inclusion $x \mapsto (x, \alpha)$. A presheaf $\mathscr{F}$ is homotopy invariant iff

$$i_0^* = i_1^* : \mathscr{F}(X \times \mathbb{A}^1) \to \mathscr{F}(X)$$

for all X. Voevodsky has shown:

Theoreme 2.3.2. *1. Let $\mathscr{F} \in \mathbf{PST}(k)$ be homotopy invariant. Then the associated Nisnevich sheaf $a_{Nis}\mathscr{F}$ is also homotopy invariant. As well as the presheaves with transvers*

$$X \mapsto \mathrm{H}^n_{Nis}(X, \mathscr{F})$$

(and we have $\mathrm{H}^n_{Nis}(X, \mathscr{F}) \cong \mathrm{H}^n_{Zar}(X, \mathscr{F})$).

2. If $a_{Nis}\mathscr{F} = 0$ then $a_{Nis}C_*\mathscr{F}$ is an acyclic complex in $\mathbf{ST}(k)_{Nis}$.

As a consequence we obtain a

Corollary 2.3.3. *Let $\mathscr{F} \in \mathbf{PST}(k)$ and denote*

$$\mathscr{K} := a_{Nis}C_*\mathscr{F}$$

the associated complex in $\mathbf{ST}(k)_{Nis}$. Then for all $X \in \mathscr{S}m/k$

$$\mathbb{H}^n_{Nis}(X, \mathscr{K}) \cong \mathbb{H}^n_{Nis}\, 8x \times \mathbb{A}^1, \mathscr{K})$$

is homotopy invariant.

PROOF: Let $\mathscr{H}^n C_* \mathscr{F}$ be the presheaf of the cohomology of $C_* \mathscr{F}$. This is in $\mathbf{PST}(k)$. A general lemma says that for all $X \in \mathscr{S}m/k$

$$\mathscr{H}^n C_* \mathscr{F}(X) \cong \mathscr{H}^n C_* \mathscr{F}(X \times \mathbb{A}^1).$$

This is very similar to results in topology. By part 1 of the previous theorem

$$\mathrm{H}^i_{Nis}(X, a_{Nis}\mathscr{H}^n C_* \mathscr{F}) \cong \mathrm{H}^i_{Nis}\left(X \times \mathbb{A}^1, a_{Nis}\mathscr{H}^n C_* \mathscr{F}\right).$$

By the hypercohomology spectral sequence for all

$$\mathrm{H}^p_{Nis}(Y, a_{Nis}\mathscr{H}^q \mathscr{K}) \Rightarrow \mathbb{H}^{p+q}_{Nis}(Y, \mathscr{K}).$$

Taking $Y = X$ or $Y = X \times \mathbb{A}^1$ in this formula shows the claim. □

3 Motivic cohomology

3.1 the category of motives

Now more generally, if $\mathscr{K}$ is a complex of $\mathbf{PST}(k)$, then the previous construction produces a double complex

$$\cdots \to C_2 \mathscr{K} \to C_1 \mathscr{K} \to C_0 \mathscr{K} \to 0$$

and its total complex satisfies for all $X \in \mathscr{S}m/k$

$$\mathbb{H}^n(X, \operatorname{tot} C_* \mathscr{K}) \cong \mathbb{H}^n(X \times \mathbb{A}^1, \operatorname{tot} C_* \mathscr{K}).$$

Definition 3.1.1. We define the category of motives $DM^{eff}_-(k)$ to be the triangulated subcategory of the derived category $D^-(\mathbf{ST}(k)_{Nis})$ consisting of the complexes $\mathscr{K}$ such that $\mathbb{H}^n(X, \mathscr{K}) \cong \mathbb{H}^n(X \times \mathbb{A}^1, \mathscr{K})$. By the above discussion (notably the second part of the theorem and the corollary), one has a functor

$$\begin{array}{rcl} D^-(\mathbf{ST}(k)_{Nis}) & \to & DM^{eff}_-(k) \\ \mathscr{K} & \mapsto & \operatorname{tot} C_* \mathscr{K} \end{array}$$

which is left adjoint to the inclusion.

3.2 The motivic complex

Definition 3.2.1. For all $X \in \mathscr{S}m/k$ we let

$$M(X) := C_* \mathbb{Z}_{tr}(X)$$

be the associated motivic complex. This defines an object in $DM^{eff}_-(k)$.

As we have seen above, for an element $x \in X(k)$ there is a split exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(X, x) \to 0$$

where $\mathbb{Z} = \mathbb{Z}((X, x)^{\wedge 0})$. In particular,

$$\mathbb{Z}_{tr}(\mathbb{G}_m) = \mathbb{Z} \oplus \mathbb{Z}_{tr}(\mathbb{G}_m, 1),$$

or more generally

$$\mathbb{Z}_{tr}(\mathbb{G}_m^r) = \bigoplus_{i=0}^{r} \left(\bigoplus_{I \subset 1, \ldots, i} \mathbb{Z}_{tr}((\mathbb{G}_m, 1)^{\wedge \# I}) \right).$$

In the following, we will omit the '1' from $\mathbb{Z}_{tr}((\mathbb{G}_m, 1))$.

Definition 3.2.2. For each $q \geqslant 0$ we define the following complex of presheaves with transfers

$$\mathbb{Z}(q) = C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q]$$

seen as a bounded above cochain complex. If A is any other abelian group, then $A(q) = \mathbb{Z}(q) \otimes A$ is another cpomplex of presheaves with transfers.

The shifting convention implies that $\mathbb{Z}(q)^i = 0$ whenever $i > q$, and the q^{th} term is $\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})$. it is in fact a complex of nisnevich sheaves over $\mathscr{S}m\,/k$. For all $X \in \mathscr{S}m\,/k$ we denote by $\mathbb{Z}(q)_X$ the restriction to the Nisnevich (ir Zariski) site of X.

We define the motivic cohomology as follows.

Definition 3.2.3. $\mathrm{H}^{i,r}(X, \mathbb{Z}) := \mathrm{H}^i_{mot}(X, \mathbb{Z}(r)) = \mathbb{H}^(_{Nis} X, \mathbb{Z}(r))$.

This is formally

$$\mathrm{Hom}_{D^-(\mathbf{ST}(k)_{Nis})}(\mathbb{Z}_{tr}(X), \mathbb{Z}(r)[i] = \mathrm{Hom}_{DM_-^{eff}(k)}(M(X), \mathbb{Z}(r)[i])$$

where the equality follows by adjunction.

Some properties:

- $\mathrm{H}^p(X, \mathbb{Z}(q)) = 0$ if $p > q + \dim X$.
- It is unknown if $\mathrm{H}^p(X, \mathbb{Z}(q)) = 0$ if $p < 0$.
- One can easily see that
$$\mathrm{H}^p(X, \mathbb{Z}(1)) = \begin{cases} \Gamma(X, \mathscr{O}_X^*) & \text{for } p = 1 \\ \mathrm{Pic}(X) & \text{for } p = 2 \\ 0 & \text{otherwise} \end{cases}$$
- The wedge product of pointed schemes induce a map of motivic complexes
$$\mathbb{Z}(m) \otimes \mathbb{Z}(n) \to \mathbb{Z}(m+n).$$
This is not totally straight forward. The construction is homotopy associative and thus for each smooth X there are pairings
$$\mathrm{H}^p(X, \mathbb{Z}(q)) \otimes \mathrm{H}^{p'}(X, \mathbb{Z}(q')) \to \mathrm{H}^{p+p'}(X, \mathbb{Z}(q+q'))$$
which are skew-symmetric for the first grading and make $\mathrm{H}^*(X, \mathbb{Z}(*))$ into an associative graded-commutative ring.

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