

# $p$ -adic variational Hodge conjecture for line bundles

Student Number Theory Seminar  
University of Utah

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This talk is a self contained study of a particular case of the  $p$ -adic variational Hodge conjecture and can be seen as independent from the rest of the seminar. It is divided into three parts. First, we will give an overview of methods from crystalline cohomology taken from the book [3] of Berthelot and Ogus. Next we will describe the crystalline Chern classes for line bundles. The talk will culminate in the proof of the  $p$ -adic variational Hodge conjecture for line bundles, given as [4, Theorem 3.8].

## 1 Rudiments of crystalline cohomology

For this section, let  $(S, I, \gamma)$  be a PD-scheme where  $p$  is nilpotent.

### 1.1 Differential calculus with divided powers

Let  $Y$  be an  $S$ -scheme which  $\gamma$  can be extended. We can form the divided power envelope  $D_{Y/S}(\nu)$  of  $Y$  in  $Y/S^{\nu+1}$ .

$$\begin{array}{ccc} & D_{Y/S}(1) & \\ & \nearrow & \downarrow \\ Y & \longrightarrow & Y \times_S Y \end{array}$$

**Definition 1.1.** A hyper PD-stratification on a  $\mathcal{O}_Y$ -module  $\mathcal{E}$  is an isomorphism

$$\varepsilon : \mathcal{D}_{Y/S}(1) \otimes \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{D}_{Y/S}(1)$$

such that

1.  $\varepsilon$  is  $\mathcal{D}_{Y/S}(1)$ -linear.
2.  $\varepsilon$  reduces to the identity modulo the associated PD-ideal  $\overline{\mathcal{I}}_{\Delta}$ .
3. It satisfies the cocycle condition.

**Definition 1.2.** A map between HPD-stratified  $\mathcal{O}_Y$ -modules is said to be horizontal if it is compatible with the stratifications

We will construct an equivalence of categories between HPD-stratifications and crystals. According to Grothendieck the idea of a crystal is that it grows and it is rigid.

**Definition 1.3.** A crystal of  $\mathcal{O}_{Y/S}$ -modules is a sheaf  $\mathcal{E}$  such that for any morphism  $u : (U', T', \delta') \rightarrow (U, T, \delta)$  in  $\text{Cris}(Y/S)$  the transition map

$$u^* \mathcal{E}_{(U, T, \delta)} \rightarrow \mathcal{E}_{(U', T', \delta')}$$

is an isomorphism.

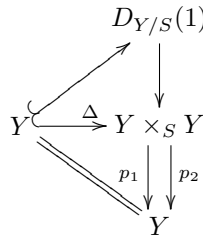
**Lemma 1.4.** *There is an equivalence of categories between crystals of  $\mathcal{O}_{Y/S}$ -modules on  $\text{Cris}(Y/S)$  with morphisms the  $\mathcal{O}_{Y/S}$ -linear maps and HPD-stratified  $\mathcal{O}_Y$ -modules with horizontal  $\mathcal{O}_Y$ -linear maps.*

PROOF: Let  $\mathcal{E}$  be a crystal. Evaluation on the trivial PD-scheme  $(Y, Y, 0)$

$$\mathcal{E} \mapsto E = \mathcal{E}_Y$$

is a functor from crystals to HPD-stratifications:

Consider the diagram

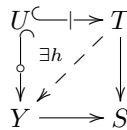


This induces morphisms

$$\mathcal{D}_{Y/S}(1) \otimes E = p_2^* E \rightarrow \mathcal{E}_{(Y, D_{Y/S}(1))} \leftarrow p_1^* E = E \otimes \mathcal{D}_{Y/S}(1)$$

which are in fact isomorphisms due to the crystalline property.

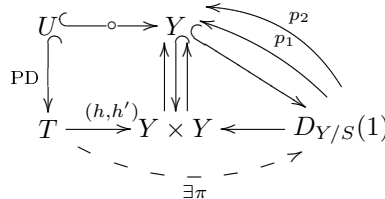
On the other hand, let  $E$  be a HPD-stratified  $\mathcal{O}_Y$ -module. Let  $(U, T, \delta) \in \text{Cris}(Y/S)$ .



As  $Y/S$  is smooth and  $U \hookrightarrow T$  a nilimmersion,  $U \rightarrow S$  lifts (locally) to  $T$ . Now we define

$$\mathcal{E}_{(U, T, \delta)} := h^* E$$

This doesn't depend on  $h$ : If  $h' : T \rightarrow Y$  is another lifting then



We have  $h = p_1 \pi$  and  $h' = p_2 \pi$  and thus

$$h^* E = \pi^* p_1^* E \cong \pi^* p_2^* E = h'^* E$$

due to the stratification condition of  $E$ .

Moreover, by construction this is a crystal (transitions are isomorphisms). □

### 1.2 The $L$ -construction of Grothendieck

Let  $Y$  be a PD-scheme where  $\gamma$  can be extended. We define a linearisation functor  $L_Y$  between the category of  $\mathcal{O}_Y$ -modules with HPD-differential operators and HPD-stratified  $\mathcal{O}_Y$ -modules with horizontal  $\mathcal{O}_Y$ -linear maps. For an  $\mathcal{O}_Y$ -module  $E$  let

$$L_Y(E) := \mathcal{D}_{Y/S}(1) \otimes E.$$

It has a canonical HPD-stratification  $\mathcal{D}_{Y/S}(1) \otimes L_Y(E) \rightarrow L_Y(E) \otimes \mathcal{D}_{Y/S}(1)$  given by

$$\begin{array}{ccccc}
 \mathcal{O}_Y \otimes E & \longrightarrow & E \otimes \mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y \otimes (E \otimes \mathcal{O}_Y) \\
 \parallel & & & & \parallel \\
 (\mathcal{O}_Y \otimes \mathcal{O}_Y) \otimes E & \longrightarrow & & \longrightarrow & (\mathcal{O}_Y \otimes \mathcal{O}_Y) \otimes E \otimes (\mathcal{O}_Y \otimes \mathcal{O}_Y) \\
 \downarrow & \nearrow & & & \downarrow \\
 (\mathcal{O}_Y \otimes \mathcal{O}_Y) \otimes (\mathcal{O}_Y \otimes \mathcal{O}_Y) \otimes E & & & & \\
 \downarrow & & & & \downarrow \\
 \mathcal{D}_{Y/S}(1) \otimes \mathcal{D}_{Y/S}(1) \otimes E & \dashrightarrow & & \dashrightarrow & \mathcal{D}_{Y/S}(1) \otimes E \otimes \mathcal{D}_{Y/S}(1)
 \end{array}$$

Let furthermore  $u : E \rightarrow F$  be a HPD-differential operator, i.e. a  $\mathcal{O}_Y$  linear map

$$\mathcal{D}_{Y/S}(1) \otimes E \rightarrow F.$$

Then  $L_Y(u)$  is to be the composition

$$L_Y(u) : \mathcal{D}_{Y/S}(1) \otimes E \xrightarrow{\partial \otimes \text{id}} \mathcal{D}_{Y/S}(1) \otimes \mathcal{D}_{Y/S}(1) \otimes E \xrightarrow{\text{id} \otimes u} \mathcal{D}_{Y/S}(1) \otimes F$$

and it is horizontal. If  $Y/S$  is smooth then  $\mathcal{D}_{Y/S}(1)$  is locally free and therefore  $L_Y$  exact.

### 1.3 The crystalline Poincaré lemma

This is an important ingredient for the proof later. Consider a closed immersion of  $S$ -schemes

$$\begin{array}{ccc}
 X & \xrightarrow{i} & Y \\
 & \searrow & \swarrow \\
 & S &
 \end{array}$$

such that  $Y/S$  is smooth and  $\gamma$  extends to  $X$ . For each  $\mathcal{O}_Y$ -modules, we set

$$L(E) := i_{\text{cris}}^* L_Y E$$

which is a crystal on  $\text{Cris}(X/S)$ . The differential operator  $d$  on the de Rham complex  $\Omega_{Y/S}$  is of order  $\leq 1$  and induces an HPD-differential operator

$$\begin{array}{ccc}
 \Omega_{Y/S}^i & \xrightarrow{d} & \Omega_{Y/S}^{i+1} \\
 \downarrow & \dashrightarrow & \downarrow \\
 \mathcal{D}_{Y/S}(1) \otimes \Omega_{Y/S}^i & & \mathcal{D}_{Y/S}(1) \otimes \Omega_{Y/S}^i \\
 \downarrow & \dashrightarrow & \downarrow \\
 \mathcal{D}_{Y/S}^1(1) \otimes \Omega_{Y/S}^i & \xleftarrow{\sim} & \mathcal{D}_{Y/S}^1(1) \otimes \Omega_{Y/S}^i
 \end{array}$$

One can show that  $d \otimes d = 0$  as HPD-differential operator (this is more than showing  $= 0$  as differential operator of degree  $\leq 2$ ) and one thereby obtains using the linearisation operator  $L$  a complex  $(L(\Omega_{Y/S}), L(d))$ .

Now the crystalline Poincaré Lemma reads:

**Theoreme 1.5.** *There is a natural quasi-isomorphism of complexes of abelian sheaves in  $(X/S)_{\text{textcris}}$*

$$\mathcal{O}_{X/S} \rightarrow L(\Omega_{Y/S})$$

or in other words the complex of crystals  $L(\Omega_{Y/S})$  is a resolution of  $\mathcal{O}_{X/S}$ .

PROOF (Sketch): There is a natural map

$$\begin{aligned} \mathcal{O}_Y &\rightarrow \mathcal{O}_Y \otimes \mathcal{O}_Y \rightarrow \mathcal{D}_{Y/S}(1) = L_Y(\mathcal{O}_Y) \\ y &\mapsto y \otimes 1 \end{aligned}$$

which is horizontal: the HPD-stratification on  $\mathcal{O}_Y$  corresponds to the identity map  $\mathcal{D}_{Y/S}(1) \rightarrow \mathcal{D}_{Y/M}(1)$  and the one on  $\mathcal{D}_{Y/S}(1)$  to the map

$$\mathcal{D}_{Y/S}(1) \otimes \mathcal{D}_{Y/S}(1) \rightarrow \mathcal{D}_{Y/S}(1) \otimes \mathcal{D}_{Y/S}(1)$$

coming from

$$a \otimes b \otimes c \otimes d \mapsto 1 \otimes d \otimes a \otimes bc.$$

In order to verify that this gives rise to a morphism of complexes one needs to check that the composition

$$\mathcal{O}_Y \rightarrow L_Y(\mathcal{O}_Y) \rightarrow L_Y(\Omega_{Y/S}^1)$$

is zero, which can be seen using local coordinates.

Because of the correspondence of stratification and crystals mentioned earlier the construction induces a map of complexes of crystals on  $Y/S$

$$\mathcal{O}_{Y/S} \rightarrow L(\Omega_{Y/S})$$

and pulling back via  $i_{\text{cris}}^*$  gives the desired morphism.

To show that it is a quasi-isomorphism is a local question on  $\text{Cris}(X/S)$ , and we may verify it on a “small enough” open  $(U, T, \delta)$  which admits a section

$$\begin{array}{ccc} U \hookrightarrow T & & \\ \downarrow \circlearrowleft & & \downarrow \exists h \\ X \hookrightarrow Y & & \end{array}$$

(due to smoothness of  $Y/S$  and  $U \hookrightarrow T$  is a nilimmersion). It is straight forward to show that

$$L(\Omega_{Y/S})_{(U,T,\delta)} \cong h^*(L_Y(\Omega_{Y/S})) = \mathcal{O}_T \otimes L_Y(\Omega_{Y/S})$$

Locally on  $Y$  the PD-envelope  $\mathcal{D}_{Y/S}(1)$  is isomorphic to a PD-algebra  $\mathcal{O}_Y \langle \xi_1 \dots \xi_n \rangle$  with  $\xi_i = 1 \otimes x_i - x_i \otimes 1$ . Therefore,  $L(\Omega_{Y/S})_{(U,T,\delta)}$  is the de Rham complex of this algebra, and it remains to show that it gives a resolution of  $\mathcal{O}_T$ .

By induction on the number of coordinates.  $n = 1$ :  $\Omega_{T/S} = [\mathcal{O}_T \xrightarrow{d} \mathcal{O}_T dx]$  and applying  $L_Y$  yields

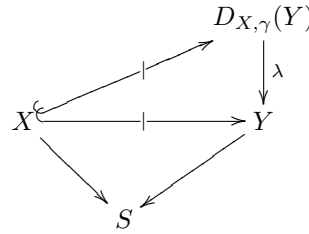
$$\begin{aligned} L_Y(\Omega_{Y/S})_T &= [\mathcal{O}_T \langle \xi \rangle \rightarrow \mathcal{O}_T \langle \xi \rangle dx \\ \xi^{[i]} &\mapsto \xi^{[i-1]} dx \end{aligned}$$

Then the map  $\mathcal{O}_T \rightarrow [\mathcal{O}_T \langle \xi \rangle \rightarrow \mathcal{O}_T \langle \xi \rangle dx]$  is a quasi-isomorphism.

The induction uses the same reasoning just adding one variable at a time. □

### 1.4 Comparison between crystalline and de Rham cohomology

Now let  $D_{X,\gamma}(Y)$  be the PD-envelope of  $X$  in  $Y$  compatible with  $\gamma$



which is canonically endowed with an integrable connection. Let  $u_{X/S} : (X/S)_{\text{cris}} \rightarrow X_{\text{zar}}$  be the projection.

**Theorem 1.6.** *There is an isomorphism*

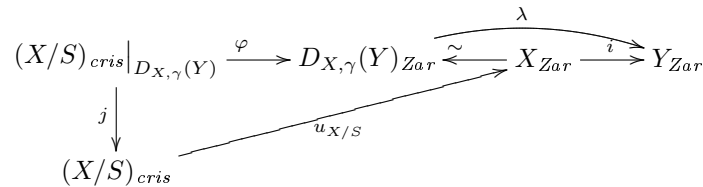
$$Ru_{X/S,*} \mathcal{O}_{X/S} \xrightarrow{\sim} \mathcal{D}_{X,\gamma}(Y) \otimes \Omega_{Y/S}$$

in the derived category  $D(X_{\text{zar}}\mathbb{Z})$ .

PROOF: To prove this, one has to apply the crystalline Poincaré Lemma on  $X_{\text{zar}}$ .

We state the following facts without proof.

**Lemma 1.7.** • *The diagram of topoi*



is commutative. The localised topos  $(X/S)_{\text{cris}}|_{D_{X,\gamma}(Y)}$  can be described as compatible families  $\{\mathcal{F}_u\}_u$  where  $u : (U, T, \delta) \rightarrow (X, D_{X,\gamma}(Y))$ .  $\varphi$  is defined by

$$\varphi_* \mathcal{F} = \mathcal{F}_{\text{id}_{D_{X,\gamma}(Y)}} \quad , \quad \phi^{-1} \mathcal{F} = \{u^{-1} \mathcal{F}\}_u$$

• Secondly, for each  $Y$ -module  $E$ ,

$$l(E) \xrightarrow{\sim} j_*(\lambda \circ \varphi)^* E = j_* \varphi^*(D_{X,\gamma}(Y) \otimes E)$$

in  $((X/S)_{\text{cris}}, \mathcal{O}_{X/S})$ .

**Lemma 1.8.** • *The functor  $j_*$  is exact.*

• *For an abelian sheaf  $\mathcal{F}$  on  $(X/S)_{\text{cris}}$ , the sheaf  $j_* \mathcal{F}$  is  $u_{X/S,*}$ -acyclic.*

Starting with the isomorphism of the crystalline Poincaré Lemma

$$\mathcal{O}_{X/S} \xrightarrow{\sim} L(\Omega_{Y/S})$$

we apply the functor  $Ru_{X/S,*} : D((X/S)_{\text{cris}}, \mathbb{Z}) \rightarrow D(X_{\text{zar}}, \mathbb{Z})$

$$Ru_{X/S,*} \mathcal{O}_{X/S} \xrightarrow{\sim} Ru_{X/S,*} L(\Omega_{Y/S}).$$

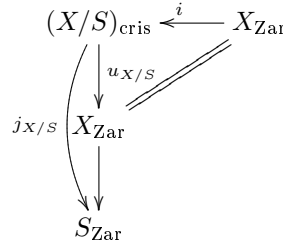
By the first lemma,  $L(\Omega_{Y/S}) \cong j_* \varphi^*(D_{X,\gamma}(Y) \otimes \Omega_{Y/S})$ . Applying  $Ru_{X/S,*}$  and using acyclicity from the second lemma, one can write

$$\begin{aligned}
 Ru_{X/S,*} L(\Omega_{Y/S}) &= u_{X/S,*} j_* \varphi^*(D_{X,\gamma}(Y) \otimes \Omega_{Y/S}) && \text{(by acyclicity)} \\
 &= \varphi_* \varphi^*(D_{X,\gamma}(Y) \otimes \Omega_{Y/S}) && \text{(by commutative of the diagram)} \\
 &= D_{X,\gamma}(Y) \otimes \Omega_{Y/S}.
 \end{aligned}$$

Putting all together gives the desired isomorphism. □

## 2 Crystalline Chern classes for line bundles

Let  $(S, I, \gamma)$  be a PD-scheme where  $p$  is nilpotent, and  $f : X \rightarrow S$  an  $S$ -scheme such that the derived powers  $\gamma$  extend.



### 2.1 Definition

We will construct a map

$$c^{\text{cris}} : R^1 f_* \mathcal{O}_X^* \rightarrow R^2 f_{X/S,*} \mathcal{O}_{X/S},$$

which induces the first Chern class map

$$c_1^{\text{cris}} : \text{Pic}(X) = H^1(X, \mathcal{O}_X^*) \rightarrow H_{\text{cris}}^2(X/S).$$

There is a canonical short exact sequence in the topos  $(X/S)_{\text{cris}}$ .

$$0 \rightarrow \mathcal{I}_{X/S} \rightarrow \mathcal{O}_{X/S} \rightarrow i_* \mathcal{O}_X \rightarrow 0$$

which has on an open  $(U, T, \delta)$  the realisation

$$0 \rightarrow (\text{Ker}(\mathcal{O}_T \rightarrow \mathcal{O}_U)) \rightarrow \mathcal{O}_T \rightarrow \mathcal{O}_U \rightarrow 0.$$

By definition the morphism of topoi

$$i = (i^*, i_*) := (u_{X/S}^!, u_{X/S}^*)$$

(it is defined this way, to provide a section for the morphism of topoi  $u_{X/S}$ , which is then justified to be called a projection). Thus in particular  $i_*$  is exact.

The multiplicative version of this sequence is given by

$$1 \rightarrow 1 + \mathcal{I}_{X/S} \rightarrow \mathcal{O}_{X/S}^* \rightarrow i_* \mathcal{O}_X^* \rightarrow 1$$

where the mod  $\mathcal{I}_{X/S}$ -map  $\mathcal{O}_{X/S}^* \rightarrow i_* \mathcal{O}_X^*$  is an epimorphism because  $\mathcal{I}_{X/S}$  is a nilideal. If we apply the functor  $f_{X/S,*}$  the associated long exact sequence of cohomology has a coboundary map

$$R^1 f_{X/S,*} i_* \mathcal{O}_X^* \xrightarrow{\partial} R^2 f_{X/S,*} (1 + \mathcal{I}_{X/S}).$$

Together with the identity  $R^1 f_* \mathcal{O}_X^* = R^1 f_{X/S,*} i_* \mathcal{O}_X^*$  due to exactness of  $i_*$ , this is the first part of the desired morphism.

Next, we take the logarithm

$$\begin{aligned}
 \log : 1 + \mathcal{I}_{X/S} &\rightarrow \mathcal{I}_{X/S} \\
 1 + x &\mapsto \sum_{m=1}^{\infty} (-1)^{m+1} (m-1)! \delta_m(x)
 \end{aligned}$$

Note that the sum is always finite as  $\mathcal{I}_{X/S}$  is a  $p^N$ -torsion ideal. By functoriality this induces a map on the derived objects

$$R^2 f_{X/S,*} (1 + \mathcal{I}_{X/S}) \xrightarrow{\log_\gamma}$$

taking also in account that  $\mathcal{I}_{X/S} \subset \mathcal{O}_{X/S}$ . Putting it all together yields

**Definition 2.1.** The crystalline Chern class is given by the composition

$$c^{\text{cris}} : R^1 f_* \mathcal{O}_X^* \xrightarrow{\partial} R^2 f_{X/S,*} (1 + \mathcal{I}_{X/S}) \xrightarrow{\log_\gamma} R^2 f_{X/S,*} \mathcal{I}_{X/S} \rightarrow R^2 f_{X/S,*} \mathcal{O}_{X/S}$$

## 2.2 Description à la de Rham

We consider a lifting situation

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{V}(I) & \hookrightarrow & S \end{array}$$

where the right vertical map is smooth and the diagram commutes. From the previous section, we see that there are isomorphisms  $Ru_{X/S*} \mathcal{O}_{X/S} \xrightarrow{\sim} \Omega_{Y/S}$  and  $Rf_{X/S*} \mathcal{O}_{X/S} \xrightarrow{\sim} Rf_* \Omega_{Y/S}$  in the derived categories  $D(X_{\text{Zar}}, \mathbb{Z})$  and  $D(S_{\text{Zar}}, \mathbb{Z})$  respectively. Therefore, we can consider the diagram

$$\begin{array}{ccc} R^1 f_* \mathcal{O}_X^* & \xrightarrow{c^{\text{cris}}} & R^2 f_{X/S*} \mathcal{O}_{X/S} \\ & \searrow & \downarrow \sim \\ & & R^2 f_* \Omega_{Y/S} \end{array}$$

We attempt now to construct this morphism.

There is an exact sequence of complexes on the Zariski topos of  $X$

$$0 \rightarrow I_{Y/S}^\bullet \rightarrow \Omega_{Y/S} \rightarrow \mathcal{O}_X \rightarrow 0$$

where

$$\begin{aligned} I_{Y/S}^\bullet &:= [I \mathcal{O}_Y \rightarrow \Omega_{Y/S}^{\geq 1}] \\ \Omega_{Y/S} &:= [\mathcal{O}_Y \rightarrow \Omega_{Y/S}^{\geq 1}] \end{aligned}$$

and the righthand members coincide. The multiplicative version

$$1 \rightarrow K_{Y/S}^* \rightarrow \Omega_{Y/S}^* \rightarrow \mathcal{O}_X^* \rightarrow 0$$

with

$$\begin{aligned} K_{Y/S}^* &:= [1 + I \mathcal{O}_Y \rightarrow \Omega_{Y/S}^{\geq 1}] \\ \Omega_{Y/S}^* &:= [\mathcal{O}_Y^* \xrightarrow{d \log} \Omega_{Y/S}^{\geq 1}] \end{aligned}$$

gives rise to the boundary map

$$\partial : R^1 f_* \mathcal{O}_X^* \rightarrow R^2 f_* K_{Y/S}^*$$

As before, taking the logarithm on the complex  $K_{Y/S}^*$  componentwise in the obvious way

$$K_{Y/S}^* \xrightarrow{\log^\bullet} I_{Y/S}^\bullet$$

yields via functoriality a morphism

$$R^2 f_* K_{Y/S}^* \rightarrow R^2 f_* I_{Y/S}^\bullet \rightarrow R^2 f_* \Omega_{Y/S}$$

and we have the following statement.

**Proposition 2.2.** *The diagram*

$$\begin{array}{ccccccc} R^1 f_* \mathcal{O}_X^* & \xrightarrow{c^{\text{cris}}} & & \rightarrow & R^2 f_{X/S*} \mathcal{O}_{X/S} & & \\ \parallel & & & & \downarrow \sim & & \\ R^1 f_* \mathcal{O}_X^* & \xrightarrow{\partial} & R^2 f_* K_{Y/S}^* & \xrightarrow{\log^\bullet} & R^2 f_* I_{Y/S}^\bullet & \longrightarrow & R^2 f_* \Omega_{Y/S} \end{array}$$

*commutes.*

This shows the claim made at the beginning.

### 3 The $p$ -adic variational Hodge conjecture for line bundles

We have now done the major part of the work and can proceed to the actual statement.

Let  $(S, I, \delta)$  be the canonical PD-structure  $(\mathrm{Spf}(W), (p), \mathrm{can})$  of the Witt vectors. We consider the lifting situation

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{V}(I) & \hookrightarrow & S \end{array}$$

where the right vertical map is a formal,  $p$ -adic and smooth morphism. The previous proposition applies in this context and the projection on the components in degree 0 give the following diagram

$$\begin{array}{ccccccc} R^1 f_* \mathcal{O}_X^* & \xrightarrow{c^{\mathrm{cris}}} & R^2 f_{X/S} \mathcal{O}_{X/S} & & & & \\ \parallel & & \downarrow \sim & & & & \\ R^1 f_* \mathcal{O}_X^* & \xrightarrow{\partial} & R^2 f_* K_{\mathcal{Y}/S}^* & \xrightarrow{\log^\bullet} & R^2 f_* I_{\mathcal{Y}/S}^\bullet & \longrightarrow & R^2 f_* \Omega_{\mathcal{Y}/S} \\ \parallel & & \downarrow & & \downarrow & & \downarrow \\ R^1 f_* \mathcal{O}_{\mathcal{Y}}^* & \longrightarrow & R^1 f_* \mathcal{O}_X^* & \xrightarrow{\partial} & R^2 f_*(1 + I \mathcal{O}_{\mathcal{Y}}) & \xrightarrow{\log} & R^2 f_*(I \mathcal{O}_{\mathcal{Y}}) & \longrightarrow & R^2 f_* \mathcal{O}_{\mathcal{Y}} \\ & & \searrow \mathrm{Ob} & & \downarrow \sim & & \downarrow \sim & & \\ & & & & R^2 f_* \mathcal{O}_{\mathcal{Y}} & & & & \end{array}$$

For  $p > 2$  the divided powers on the ideal  $(p) \subset W$  are  $p$ -adically nilpotent, which allows the definition of the exponential map

$$I \mathcal{O}_{\mathcal{Y}} \ni y \mapsto \sum \bar{\gamma}_n(y) \in 1 + I \mathcal{O}_{\mathcal{Y}}$$

and we may assume that  $\log : 1 + I \mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{O}_{\mathcal{Y}}$  is an isomorphism. The composition

$$\mathrm{Ob} := \log \circ \partial$$

can therefore be taken as the obstruction to lift an element  $L \in R^1 f_* \mathcal{O}_X^*$  to a formal class in  $R^1 f_* \mathcal{O}_{\mathcal{Y}}^*$ . The kernel of the map  $R^2 f_* \Omega_{\mathcal{Y}/S} \rightarrow R^2 f_* \mathcal{O}_{\mathcal{Y}}$  is precisely the first piece of the filtration  $F^1 R^2 f_* \Omega_{\mathcal{Y}/S}$ . We conclude from this:

- If  $L \in R^1 f_* \mathcal{O}_X^*$  can be lifted to  $R^1 f_* \mathcal{O}_{\mathcal{Y}}^*$ , then  $c^{\mathrm{cris}}(L) \in F^1 R^2 f_* \Omega_{\mathcal{Y}/S}$ .
- If  $L \in R^1 f_* \mathcal{O}_X^*$  satisfies  $c^{\mathrm{cris}}(L) \in F^1 R^2 f_* \Omega_{\mathcal{Y}/S}$ , then it can be lifted in  $R^1 f_* \mathcal{O}_{\mathcal{Y}}^*$ .

Thus we have seen:

$$L \in \mathrm{Pic}(X) \otimes \mathbb{Q} \text{ lifts to } \mathrm{Pic}(\mathcal{Y}) \otimes \mathbb{Q} \text{ if and only if } c^{\mathrm{cris}}(L) \in F^1 H_{dR}^2(\mathcal{Y}/W) \otimes \mathbb{Q}.$$

If  $\mathcal{Y}$  is the  $p$ -adic completion of a proper  $W$ -scheme, the existence theorem of Grothendieck allows us to algebraise. In that case the aforementioned statement proves the  $p$ -adic variational Hodge conjecture by Fontaine and Messing.

*Remark 3.1.* This statement is also true for  $p = 2$ . It can also be extended to the case when  $W$  is a discrete valuation ring of mixed characteristic (cf. [4]).



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