p-adic variational Hodge conjecture for line bundles

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This talk is a self contained study of a particular case of the p-adic variational Hodge conjecture and can be seen as independent from the rest of the seminar. It is devided into three parts. First, we will give an overview of methods from crystalline cohomology taken from the book [3] of Berthelot and Ogus. Next we will describe the crystalline Chern classes for line bundles. The talk will culminate in the proof ofthe p-adic variational Hidge conjecture for line bundles, given as [4, Theorem 3.8].

1 Rudiments of crystalline cohomology

For this section, let (S, I, γ) be a PD-scheme where p is nilpotent.

1.1 Differential calculus with devided powers

Let Y be an S-schemeon which γ can be extended. We can form Th devided power envelope $D_{Y/S}(\nu)$ of Y in $Y/S^{\nu+1}$.

$$\begin{tikzcd} & D_{Y/S}(1) \arrow[d] \\ Y \arrow[ur, hook] \arrow[r, hook] & Y \times_S Y \end{tikzcd}$$

Definition 1.1. A hyper PD-stratification on a $\mathscr{O}_Y$-module $\mathscr{E}$ is an isomorphism

$$\varepsilon : \mathscr{D}_{Y/S}(1) \otimes \mathscr{E} \to \mathscr{E} \otimes \mathscr{D}_{Y/S}(1)$$

such that

1. ε is $\mathscr{D}_{Y/S}(1)$-linear.

2. ε reduces to the identity modulo the associated PD-ideal $\overline{\mathscr{I}}_\Delta$.

3. It satisfies the cocycle condition.

Definition 1.2. A map between HPD-stratified $\mathscr{O}_Y$-modules is said to be horizontal if it is compatible with the stratifications

We will construct an equivalence of categories between HPD-stratifications and crystals. According to Grothendieck the idea of a crystal is that it grows and it is rigid.

Definition 1.3. A crystal of $\mathscr{O}_{Y/S}$-modules is a sheaf $\mathscr{E}$ such that for any morphism $u : (U', T', \delta') \to (U, T, \delta)$ in $\mathrm{Cris}(Y/S)$ the transition map

$$u^* \mathscr{E}_{(U,T,\delta)} \to \mathscr{E}_{(U',T',\delta')}$$

is an isomorphism.

Lemma 1.4. *There is an equivalence of categories between crystals of $\mathscr{O}_{Y/S}$-modules on* $\mathrm{Cris}(Y/S)$ *with morphisms the $\mathscr{O}_{Y/S}$-linear maps and HPD-stratified $\mathscr{O}_Y$-modules with horizontal $\mathscr{O}_Y$-linear maps.*

PROOF: Let $\mathscr{E}$ be a crystal. Evaluation on the trivial PD-scheme $(Y, Y, 0)$

$$\mathscr{E} \mapsto E = \mathscr{E}_Y$$

is a functor from crystals to HPD-stratifications:
Consider the diagram

$$\begin{tikzcd}
 & D_{Y/S}(1) \arrow[d] \\
Y \arrow[ur, hook] \arrow[r, "\Delta"] \arrow[dr, equal] & Y \times_S Y \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
 & Y
\end{tikzcd}$$

This induces morphisms

$$\mathscr{D}_{Y/S}(1) \otimes E = p_2^* E \to \mathscr{E}_{(Y, D_{Y/S}(1)} \leftarrow p_1^* E = E \otimes \otimes \mathscr{D}_{Y/S}(1)$$

which are in fact isomorphisms due to the crystalline property.
On the other hand, let E be a HPD-stratified $\mathscr{O}_Y$-module. Let $(U, T, \delta) \in \mathrm{Cris}(Y/S)$.

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, hook] & T \arrow[d] \arrow[dl, dashed, "\exists h"'] \\
Y \arrow[r] & S
\end{tikzcd}$$

As Y/S is smooth and $U \hookrightarrow T$ a nilimmersion, $U \to S$ lifts (locally) to T. Now we define

$$\mathscr{E}_{(U,T,\delta)} := h^* E$$

This doesn't depend on h: If $h' : T \to Y$ is another lifting then

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, hook, "\mathrm{PD}"'] & Y \arrow[d, shift left] \arrow[dr, hook] & \\
T \arrow[r, "{(h,h')}"] \arrow[rr, dashed, bend right, "\exists \pi"'] & Y \times Y \arrow[u, shift left] \arrow[u, shift left=3] & D_{Y/S}(1) \arrow[l] \arrow[ul, bend right, "p_1"'] \arrow[ul, bend right=50, "p_2"']
\end{tikzcd}$$

We have $h = p_1 \pi$ and $h' = p_2 \pi$ and thus

$$h^* E = \pi^* p_1^* E \cong \pi^* p_2^* E = h'^* E$$

due to the stratification condition of E.
Moreover, by construction this is a crystal (transitions are isomorphisms). $\square$

1.2 The L-construction of Grothendieck

Let Y be a PD-scheme where γ can be extended. We define a linearisation functor L_Y between the category of $\mathscr{O}_Y$-modules with HPD-differential operators and HPD-stratified $\mathscr{O}_Y$-modules with horizontal $\mathscr{O}_Y$-linear maps. For an $\mathscr{O}_Y$-module E let

$$L_Y(E) := \mathscr{D}_{Y/S}(1) \otimes E.$$

It has a canonical HPD-stratification $\mathscr{D}_{Y/S}(1) \otimes L_Y(E) \to L_Y(E) \otimes \mathscr{D}_{Y/S}(1)$ given by

$$\begin{tikzcd}
\mathscr{O}_Y \otimes E \arrow[r] \arrow[d, equal] & E \otimes \mathscr{O}_Y \arrow[r] & \mathscr{O}_Y \otimes (E \otimes \mathscr{O}_Y) \arrow[d, equal] \\
(\mathscr{O}_Y \otimes \mathscr{O}_Y) \otimes E \arrow[rr] \arrow[d] & & (\mathscr{O}_Y \otimes \mathscr{O}_Y) \otimes E \otimes (\mathscr{O}_Y \otimes \mathscr{O}_Y) \arrow[dd] \\
(\mathscr{O}_Y \otimes \mathscr{O}_Y) \otimes (\mathscr{O}_Y \otimes \mathscr{O}_Y) \otimes E \arrow[urr] \arrow[d] & & \\
\mathscr{D}_{Y/S}(1) \otimes \mathscr{D}_{Y/S}(1) \otimes E \arrow[rr, dashed] & & \mathscr{D}_{Y/S}(1) \otimes E \otimes \mathscr{D}_{Y/S}(1)
\end{tikzcd}$$

Let furthermore $u : E \to F$ be a HPD-differential operator, i.e. a $\mathscr{O}_Y$ linear map

$$\mathscr{D}_{Y/S}(1) \otimes E \to F.$$

Then $L_Y(u)$ is to be the composition

$$L_Y(u) : \mathscr{D}_{Y/S}(1) \otimes E \xrightarrow{\partial \otimes \mathrm{id}} \mathscr{D}_{Y/S}(1) \otimes \mathscr{D}_{Y/S}(1) \otimes E \xrightarrow{\mathrm{id} \otimes u} \mathscr{D}_{Y/S}(1) \otimes F$$

and it is horizontal. If Y/S is smooth then $\mathscr{D}_{Y/S}(1)$ is locally free and therefore L_Y exact.

1.3 The crystalline Poincaré lemma

This is an important ingredient for the proof later. Consider a closed immersion of S-schemes

$$\begin{tikzcd}
X \arrow[rr, hook, "i"] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

such that Y/S is smooth and γ extends to X. For each $\mathscr{O}_Y$-modules, we set

$$L(E) := i^*_{\mathrm{cris}} L_Y E$$

which is a crystal on $\mathrm{Cris}(X/S)$. The differential operator d on the de Rham complex $\Omega_{Y/S}$ is of order $\leqslant 1$ and induces an HPD-differential operator

$$\begin{tikzcd}
& \Omega^i_{Y/S} \arrow[r, "d"] \arrow[dd] & \Omega^{i+1}_{Y/S} \\
\mathscr{D}_{Y/S}(1) \otimes \Omega^i_{Y/S} \arrow[urr, dashed] \arrow[d] & & \\
\mathscr{D}^1_{Y/S}(1) \otimes \Omega^i_{Y/S} & \mathscr{D}^1_{Y/S}(1) \otimes \Omega^i_{Y/S} \arrow[l, "\sim"'] \arrow[uur] &
\end{tikzcd}$$

One can show that $d \otimes d = 0$ as HPD-differential operator (this is more that showing $= 0$ as differential operator of degree $\leqslant 2$) and one thereby obtains using the linearisation operator L a complex $(L(\Omega_{Y/S}), L(d))$.
Now the crystalline Poincaré Lemma reads:

Theoreme 1.5. *There is a natural quasi-isomorphism of complexes of abelian sheaves in* $(X/S)_{textcris}$

$$\mathscr{O}_{X/S} \to L(\Omega_Y/S)$$

or in other words the complex of crystals $L(\Omega_Y/S)$ *is a resolution of* $\mathscr{O}_{X/S}$.

Proof (Sketch): There is a natural map

$$\begin{array}{rcl} \mathscr{O}_Y & \to & \mathscr{O}_Y \otimes \mathscr{O}_Y \to \mathscr{D}_{Y/S}(1) = L_Y(\mathscr{O}_Y) \\ y & \mapsto & y \otimes 1 \end{array}$$

which is horizontal: the HPD-stratification on $\mathscr{O}_Y$ corresponds to the identity map $\mathscr{D}_{Y/S}(1) \to \mathscr{D}_{Y/M}(1)$ and the one on $\mathscr{D}_{Y/S}(1)$ to the map

$$\mathscr{D}_{Y/S}(1) \otimes \mathscr{D}_{Y/S}(1) \to \mathscr{D}_{Y/S}(1) \otimes \mathscr{D}_{Y/S}(1)$$

coming from

$$a \otimes b \otimes c \otimes d \mapsto 1 \otimes d \otimes a \otimes bc.$$

In order to verify that this gives rise to a morphism of complexesne needs to check that the composition

$$\mathscr{O}_Y \to L_Y(\mathscr{O}_Y) \to L_Y(\Omega^1_{Y/S})$$

is zero, which can be seenusing local coordinates.
Because of the correspondence of stratification and crystals mentioned earlier the construction induces a map of complexes of crystals on Y/S

$$\mathscr{O}_{Y/S} \to L(\Omega_{Y/S})$$

and puling back via i^*_{cris} gives the desired morphism.
To show that it is a quasi-isomorphism is a local question on $\mathrm{Cris}(X/S)$, and we may verify it on a "small enough" open (U, T, δ) which admits a section

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook] & T \arrow[d, dashed, "\exists h"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

(due to smoothness of Y/S and $U \hookrightarrow T$ is a nilimmersion). It is straight forward to show that

$$L(\Omega_{Y/S})_{(U,T,\delta)} \cong h^*(L_Y(\Omega_{Y/S})) = \mathscr{O}_T \otimes L_Y(\Omega_{Y/S})$$

Locally on Y the PD-envelope $\mathscr{D}_{Y/S}(1)$ is isomorphic to a PD-algebra $\mathscr{O}_Y\langle \xi_1 \dots \xi_n \rangle$ with $\xi_i = 1 \otimes x_i - x_i \otimes 1$.
Therefore, $L(\Omega_{Y/S})_{(U,T,\delta)}$ is the de Rham complex of this algebra, and it remains to show that it gives a resolution of $\mathscr{O}_T$.
By induction on the number of coordinates. $n = 1$: $\Omega_{T/S} = [\mathscr{O}_T \xrightarrow{d} \mathscr{O}_T\, dx]$ and applying L_Y yiealds

$$\begin{array}{rcl} L_Y(\Omega_{Y/S})_T = [\mathscr{O}_T\langle \xi \rangle & \to & \mathscr{O}_T\langle \xi \rangle dx \\ \xi^{[i]} & \mapsto & \xi^{[i-1]} dx \end{array}$$

Then the map $\mathscr{O}_T \to [\mathscr{O}_T\langle \xi \rangle \to \mathscr{O}_T\langle \xi \rangle dx]$ is a quasi-isomorphism.
The induction uses the same resoning jsut adding one variable at a time. $\square$

1.4 Comparison between crystalline and de Rham cohomology

Now let $D_{X,\gamma}(Y)$ be the PD-envelope of X in Y compatible with γ

$$\begin{tikzcd}
 & & D_{X,\gamma}(Y) \arrow[d, "\lambda"] \\
X \arrow[urr, hook] \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
 & S &
\end{tikzcd}$$

which is canonically endowed with an integrable connection. Let $u_{X/S} : (X/S)_{\text{cris}} \to X_{\text{zar}}$ be the projection.

Theoreme 1.6. *There is an isomorphism*

$$Ru_{X/S,*}\mathscr{O}_{X/S} \xrightarrow{\sim} \mathscr{D}_{X,\gamma}(Y) \otimes \Omega_{Y/S}$$

in the derrived category $D(X_{Zar}\mathbb{Z})$.

PROOF: To prove this, one has to apply the crystalline Poincaré Lemma on X_{Zar}.
We state the following facts without proof.

Lemma 1.7. • *The diagram of topoi*

$$\begin{tikzcd}
(X/S)_{cris}\big|_{D_{X,\gamma}(Y)} \arrow[r, "\varphi"] \arrow[d, "j"] & D_{X,\gamma}(Y)_{Zar} \arrow[rr, bend left, "\lambda"] & X_{Zar} \arrow[l, "\sim"'] \arrow[r, "\iota"] & Y_{Zar} \\
(X/S)_{cris} \arrow[urr, "u_{X/S}"'] & & &
\end{tikzcd}$$

is commutative. The localised topos $(X/S)_{cris}\big|_{D_{X,\gamma}(Y)}$ can be described as compatible families $\{\mathscr{F}_u\}_u$ where $u : (U,T,\delta) \to (X, D_{X,\gamma}(Y)$. φ is defined by

$$\varphi_*\mathscr{F} = \mathscr{F}_{\mathrm{id}_{D_{X,\gamma}(Y)}} \quad , \quad \phi^{-1}\mathscr{F} = \{u^{-1}\mathscr{F}\}_u$$

- *Secondly, for each $_Y$-module E,*

$$l(E) \xrightarrow{\sim} j_*(\lambda \circ \varphi)^* E = j_*\varphi^*(D_{X,\gamma}(Y) \otimes E)$$

in $((X/S)_{cris}, \mathscr{O}_{X/S})$.

Lemma 1.8. • *The functor j_* is exact.*

- *For an abelian sheaf $\mathscr{F}$ on $(X/S)_{cris}$, the sheaf $j_*\mathscr{F}$ is $u_{X/S,*}$-acyclic.*

Starting with the isomorphism of the crystalline Poincaré Lemma

$$\mathscr{O}_{X/S} \xrightarrow{\sim} L(\Omega_{Y/S}$$

we apply the functor $Ru_{X/S,*} : D((X/S)_{\text{cris}}, \mathbb{Z}) \to D(X_{\text{Zar}}, \mathbb{Z})$

$$Ru_{X/S,*}\mathscr{O}_{X/S} \xrightarrow{\sim} Ru_{X/S,*}L(\Omega_{Y/S}).$$

By the first lemma, $L(\Omega_{Y/S}) \cong j_*\varphi^*(D_{X,\gamma}(Y) \otimes \Omega_{Y/S})$. Applying $Ru_{X/S,*}$ and using acyclicity from the second lemma, one can write

$$\begin{aligned}
Ru_{X/S,*}L(\Omega_{Y/S}) &= u_{X/S,*}j_*\varphi^*(D_{X,\gamma}(Y) \otimes \Omega_{Y/S}) && \text{(by acyclicity)} \\
&= \varphi_*\varphi^*(D_{X,\gamma}(Y) \otimes \Omega_{Y/S}) && \text{(by commutative of the diagram)} \\
&= D_{X,\gamma}(Y) \otimes \Omega_{Y/S}.
\end{aligned}$$

Putting all together gives the desired isomorphism. □

2 Crystalline Chern classes for line bundles

Let (S, I, γ) be a PD-scheme where p is nilpotent, and $f : X \to S$ an S-scheme such that the devided powers γ extend.

$$\begin{tikzcd}
(X/S)_{\mathrm{cris}} \arrow[d, "u_{X/S}"] \arrow[dd, bend right, "j_{X/S}"'] & X_{\mathrm{Zar}} \arrow[l, "i"'] \arrow[dl, equal] \\
X_{\mathrm{Zar}} \arrow[d] & \\
S_{\mathrm{Zar}} &
\end{tikzcd}$$

2.1 Definition

We will construct a map

$$c^{\mathrm{cris}} : R^1 f_* \mathscr{O}_X^* \to R^2 f_{X/S,*} \mathscr{O}_{X/S},$$

which induces the first Chern class map

$$c_1^{\mathrm{cris}} : \mathrm{Pic}(X) = \mathrm{H}^1(X, \mathscr{O}^*) \to \mathrm{H}^2_{\mathrm{cris}}(X/S).$$

There is a canonical short exact sequence in the topos $(X/S)_{\mathrm{cris}}$.

$$0 \to \mathscr{I}_{X/S} \to \mathscr{O}_{X/S} \to i_* \mathscr{O}_X \to 0$$

which has on an open (U, T, δ) the realisation

$$0 \to (\mathrm{Ker}(\mathscr{O}_T \to \mathscr{O}_U)) \to \mathscr{O}_T \to \mathscr{O}_U \to 0.$$

By definition the morphism of topoi

$$i = (i^*, i_*) := (u_{X/S!}, u^*_{X/S})$$

(it is defined this way, to provide a section for the morphism of topoi $u_{X/S}$, which is then justified to be called a projeciton). Thus in particular i_* is exact.
The multiplicative version of this sequence is given by

$$1 \to 1 + \mathscr{I}_{X/S} \to \mathscr{O}^*_{X/S} \to i_* \mathscr{O}^*_X \to 1$$

where the mod $\mathscr{I}_{X/S}$-map $\mathscr{O}^*_{X/S} \to i_* \mathscr{O}^*_X$ is an epimorphism because $\mathscr{I}_{X/S}$ is a nilideal. If we apply the functor $f_{X/S*}$ the associated long exact sequence of cohomology has a coboundary map

$$R^1 f_{X/S*} i_* \mathscr{O}^*_X \xrightarrow{\partial} R^2 f_{X/S*}(1 + \mathscr{I}_{X/S}).$$

Together with the identity $R^1 f_* \mathscr{O}^*_X = R^1 f_{X/S*} i_* \mathscr{O}^*_X$ due to exactness of i_*, this is the first part of the desired morphism.
Next, we take the logarithm

$$\begin{aligned} \log : 1 + \mathscr{I}_{X/S} &\to \mathscr{I}_{X/S} \\ 1 + x &\mapsto \sum_{m=1}^{\infty} (-1)^{m+1} (m-1)! \delta_n(x) \end{aligned}$$

Note that the sum is always finite as $\mathscr{I}_{X/S}$ is a p^N-torsion ideal. By fonctoriality this induces a map on the derifed objects

$$R^2 f_{X/S,*}(1 + \mathscr{I}_{X/S}) \xrightarrow{\log}$$

taking also in account that $\mathscr{I}_{X/S} \subset \mathscr{O}_{X/S}$. Putting it all together yields

Definition 2.1. The crystalline Chern class is given by the composition

$$c^{\mathrm{cris}} : R^1 f_* \mathscr{O}^*_X \xrightarrow{\partial} R^2 f_{X/S*}(1 + \mathscr{I}_{X/S}) \xrightarrow{\log} R^2 f_{X/S*} \mathscr{I}_{X/S} \to R^2 f_{X/S*} \mathscr{O}_{X/S}$$

2.2 Description à la de Rham

We consider a lifting situation

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ \mathcal{V}(I) \arrow[r, hook] & S \end{tikzcd}$$

where the right vertical map is smooth and the diagram commutes. From the previous section, we see that there are isomorphisms $Ru_{X/S*}\mathscr{O}_{X/S} \xrightarrow{\sim} \Omega_{Y/S}$ and $Rf_{X/S*}\mathscr{O}_{X/S} \xrightarrow{\sim} Rf_*\Omega_{Y/S}$ in the derived categories $D(X_{\mathrm{Zar}}, \mathbb{Z})$ and $D(S_{\mathrm{Zar}}, \mathbb{Z})$ respectively. Therefore, we can consider the diagram

$$\begin{tikzcd} R^1 f_* \mathscr{O}_X^* \arrow[r, "c^{\mathrm{cris}}"] \arrow[dr, dashed] & R^2 f_{X/S*}\mathscr{O}_{X/S} \arrow[d, "\sim"] \\ & R^2 f_* \Omega_{Y/S} \end{tikzcd}$$

We attempt now to construct this morphism.
There is an exact sequence of complexes on the Zariski topos of X

$$0 \to I^\bullet_{Y/S} \to \Omega_{Y/S} \to \mathscr{O}_X \to 0$$

where

$$\begin{aligned} I^\bullet_{Y/S} &:= \left[I\,\mathscr{O}_Y \to \Omega^{\geqslant 1}_{Y/S} \right] \\ \Omega_{Y/S} &:= \left[\mathscr{O}_Y \to \Omega^{\geqslant 1}_{Y/S} \right] \end{aligned}$$

and the righthand members coincide. The multiplicative version

$$1 \to K^*_{Y/S} \to \Omega^*_{Y/S} \to \mathscr{O}_X^* \to 0$$

with

$$\begin{aligned} K^*_{Y/S} &:= \left[1 + I\,\mathscr{O}_Y \to \Omega^{\geqslant 1}_{Y/S} \right] \\ \Omega^*_{Y/S} &:= \left[\mathscr{O}_Y^* \xrightarrow{d\log} \Omega^{\geqslant 1}_{Y/S} \right] \end{aligned}$$

gives rise to the boundary map

$$\partial : R^1 f_* \mathscr{O}_X^* \to R^2 f_* K^*_{Y/S}.$$

As before, taking the logarithm on the complex $K^*_{Y/S}$ componentwise in the obvious way

$$K^*_{Y/S} \xrightarrow{\log^\bullet} I^\bullet_{Y/S}$$

yields via functoriality a morphism

$$R^2 f_* K^*_{Y/S} \to R^2 f_* I^\bullet_{Y/S} \to R^2 f_* \Omega_{Y/S}$$

and we have the following statement.

Proposition 2.2. *The diagram*

$$\begin{tikzcd} R^1 f_* \mathscr{O}_X^* \arrow[rrr, "c^{cris}"] \arrow[d, equal] & & & R^2 f_{X/S*}\mathscr{O}_{X/S} \arrow[d, "\sim"] \\ R^1 f_* \mathscr{O}_X^* \arrow[r, "\partial"] & R^2 f_* K^*_{Y/S} \arrow[r, "\log^\bullet"] & R^2 f_* I^\bullet_{Y/S} \arrow[r] & R^2 f_* \Omega_{Y/S} \end{tikzcd}$$

commutes.

This shows the claim made at the beginning.

3 The p-adic variational Hodge conjecture for line bundles

We have now done the major part of the work and can procede to the actual statement.
Let (S, I, δ) be the canonical PD-structure $(\mathrm{Spf}(W), (p), \mathrm{can})$ of the Witt vectors. We consider the lifting situation

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathcal{V}(I) \arrow[r, hook] & S \end{tikzcd}$$

where the right vertical map is a formal, p-adic and smooth morphism. The previous proposition applies in this context and the projection on the components in degree 0 give the following diagram

$$\begin{tikzcd} & R^1 f_* \mathscr{O}_X^* \arrow[rrr, "c^{\mathrm{cris}}"] \arrow[d, equal] & & & R^2 f_{X/S*} \mathscr{O}_{X/S} \arrow[d, "\sim"] \\ & R^1 f_* \mathscr{O}_X^* \arrow[r, "\partial"] \arrow[d, equal] & R^2 f_* K^*_{\mathscr{Y}/S} \arrow[r, "\log^\bullet"] \arrow[d] & R^2 f_* I^\bullet_{\mathscr{Y}/S} \arrow[r] \arrow[d] & R^2 f_* \Omega_{\mathscr{Y}/S} \arrow[d] \\ R^1 f_* \mathscr{O}_{\mathscr{Y}}^* \arrow[r] & R^1 f_* \mathscr{O}_X^* \arrow[r, "\partial"] \arrow[rr, bend right, "\mathrm{Ob}"'] & R^2 f_*(1 + I \mathscr{O}_{\mathscr{Y}}) \arrow[r, "\log", "\sim"'] & R^2 f_*(I \mathscr{O}_{\mathscr{Y}}) \arrow[r] \arrow[d, "\sim"] & R^2 f_* \mathscr{O}_{\mathscr{Y}} \\ & & & R^2 f_* \mathscr{O}_{\mathscr{Y}} \arrow[ur] & \end{tikzcd}$$

For $p > 2$ the devided powers on the ideal $(p) \subset W$ are p-adically nilpotent, which allows the definition of the exponential map

$$I\,\mathscr{O}_{\mathscr{Y}} \ni y \mapsto \sum \overline{\gamma}_n(y) \in 1 + I\,\mathscr{O}_{\mathscr{Y}}$$

and we may assume that $\log : 1 + I\,\mathscr{O}_{\mathscr{Y}} \to \mathscr{O}_{\mathscr{Y}}$ is an isomorphism. The composition

$$\mathrm{Ob} := \log \circ \partial$$

can therefore be taken as the obstruction to lift an element $L \in R^1 f_* \mathscr{O}_X^*$ to a formal class in $R^1 f_* \mathscr{O}_{\mathscr{Y}}^*$. The kernel of the map $R^2 f_* \Omega_{\mathscr{Y}/S} \to R^2 f_* \mathscr{O}_{\mathscr{Y}}$ is precisely the first pieace of the filtration $F^1 R^2 f_* \Omega_{\mathscr{Y}/S}$. We conclude from this:

- If $L \in R^1 f_* \mathscr{O}_X^*$ can be lifted to $R^1 f_* \mathscr{O}_{\mathscr{Y}}^*$, then $c^{\mathrm{cris}}(L) \in F^1 R^2 f_* \Omega_{\mathscr{Y}/S}$.
- If $L \in R^1 f_* \mathscr{O}_X^*$ satisfies $c^{\mathrm{cris}}(L) \in F^1 R^2 f_* \Omega_{\mathscr{Y}/S}$, then it can be lifted in $R^1 f_* \mathscr{O}_{\mathscr{Y}}^*$.

Thus we have seen:
$L \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ **lifts to** $\mathrm{Pic}(\mathscr{Y}) \otimes \mathbb{Q}$ **if and only if** $c^{\mathbf{cris}}(L) \in F^1\, \mathrm{H}^2_{dR}(\mathscr{Y}/W) \otimes \mathbb{Q}$.
If $\mathscr{Y}$ is the p-adic completion of a proper W-scheme, the existence theorem of Grothendieck allows us to algebrais. In that case the aforementioned statement proves the p-adic variational Hodge conjecture by Fontaine and Messing.

Remark 3.1. This statement is also true for $p = 2$. It can also be extended to the case when W is a discrete valuation ring of mixed characteristic (cf. [4]).

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