p-adic deformation of algebraic cycle classes after S. Bloch, H. Esnault and M.Kerz

27th January 2013

The article [5] studies p-adic deformation properties of algebraic cycle classes modulo rational equivalences. A principal motivation for this is to construct new "interesting" algebraic cycles.

1 Approach

The authors suggest to do this in two steps.

1. Study formal deformations to infinitesimal thickenings.
2. Algebraise these deformations.

We will see, what is meant by this.

The setting will be: deformation of cycles in the p-adic direction for schemes X/V over a complete p-adic discrete valuation ring. The authors note that the precise geometric nature of the variety is of a lesser concern, as the problem turns out to be of deeper cohomological and K-theoretic nature, related to p-adic Hodge theory.

2 Motivation: the characteristic 0 case

An early observation made by Grothendieck: the deformation of the Picard group can be described in terms of Hodge theoretic data via the first Chern class. This is called the variational Hodge conjecture, since Grothendieck points out that it follows from the Hodge conjecture [12, Note 13].

Setting: k field of characteristic 0, $S = k[\![t]\!]$, X/S smooth and projective, $X_n \hookrightarrow X$ the closed immersion given by the ideal generated by t^n. There is a standard connection, the Gauß-Manin connection

$$\nabla : \mathrm{H}^i_{\mathrm{dR}}(X/S) \to \widehat{\Omega}^1_{S/k} \hat{\otimes} \mathrm{H}^i_{\mathrm{dR}}(X/S)$$

which is trivialisable over S. It gives an isomorphism

$$\Phi : \mathrm{H}^i_{\mathrm{dR}}(X/S)^{\nabla} \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X_1/k).$$

However, in general not an isomorphism on the Hodge filtration! This failure can be expressed by an exact obstruction sequence

$$\mathrm{Pic}(X_n) \to \mathrm{Pic}(X_1) \xrightarrow{\mathrm{Ob}} \mathrm{H}^2(X_1, \mathscr{O}_{X_1}).$$

Note that $\mathrm{H}^2(X_1, \mathscr{O}) = \mathrm{H}^2(X_1, \Omega^0) = \mathrm{H}^2_{\mathrm{dR}}(X_1/S)/F^1\,\mathrm{H}^2_{\mathrm{dR}}(X_1/S)$ by the Hodge-de Rham spectral sequence. So how the Picard group deforms can be expressed by the behavior of the morphism Ob. If it is trivial (or even zero for one element ξ_1) then an element $\xi_1 \in \mathrm{Pic}(X_1)$ can be lifted to elements $\xi_n \in \mathrm{Pic}(X_n)$ and ultimately to $\mathrm{Pic}(X)$. This is the line-bundle case of the following conjecture:

Conjecture 2.0.1. *For $\xi_1 \in K_0(X_1)_{\mathbb{Q}}$ such that*

$$\Phi^{-1} \circ \mathrm{ch}(\xi_1) \in \oplus_r F^r\,\mathrm{H}^{2r}_{dR}(X/S)$$

there is $\xi \in K_0(X)_{\mathbb{Q}}$ such that $\mathrm{ch}(\xi\big|_{X_1}) = \mathrm{ch}(\xi_1) \in \mathrm{H}^{2r}_{dR}(X_1/k)$.

3 The p-adic analogue

The setting now is: k a field of characteristic $p > 0$, $S = \operatorname{Spec} W(k)$, X/S smooth and projective, $S_n = \operatorname{Spec} W_n(k)$ such that X_n/S_n is the closed immersion defined by the ideal (p^n). The isomophism Φ of the 0-characteristic case is replaced by Berthelot's comparison isomorphism

$$\Phi : \mathrm{H}^i_{\mathrm{dR}}(X/W) \xrightarrow{\sim} \mathrm{H}^i\,\mathrm{cris}(X_1/W)$$

and the Chern character used here is the crystalline Chern character

$$\mathrm{ch} : K_0(X_1)_{\mathbb{Q}} \to \oplus_r \mathrm{H}^{2r}_{\mathrm{cris}}(X_1/W)_K.$$

Definition 3.0.2. $\xi_1 \in K_0(X_1)_{\mathbb{Q}}$ is of Hodge type if

$$\Phi^{-1} \circ \mathrm{ch}(\xi) \in \oplus_r F^r\, \mathrm{H}^{2r}_{\mathrm{dR}}(X_1/W).$$

Hence we have the following

Conjecture 3.0.3. *For $\xi \in K_0(X_1)_{\mathbb{Q}}$ the following are equivalent*

1. *ξ is of Hodge type.*
2. *There is a class $\xi \in K_0(X)_{\mathbb{Q}}$ such that $\mathrm{ch}(\xi\big|_{X_1}) = \mathrm{ch}(\xi_1) \in \mathrm{H}^{2r}_{cris}(X_1/W)_K$.*

4 The line-bundle case

The conjecture is known only in the case when $\xi_1 \in \mathrm{Pic}(X_1)$. The proof is in two steps as mentioned above: a deformation part and an algebraisation part.

Step 1. There is a short exact sequence

$$1 \to (1 + p\,\mathscr{O}_{X_n}) \to \mathscr{O}^*_{X_n} \to \mathscr{O}^*_{X_1} \to 1.$$

This induces a long exact sequence, and if we conbine the coboundary morhism with the p-adic logarithm we obtain an exact sequence

$$\varprojlim \mathrm{Pic}(X_n) \to \mathrm{Pic}(X_1) \xrightarrow{\mathrm{Ob}} \mathrm{H}^2(X, p\,\mathscr{O}) \tag{1}$$

which will play the role of the obstruction as above.

In fact, one shows that $\xi_1 \in \mathrm{Pic}(X_1)$ is of Hodge type iff $\mathrm{Ob}(\xi_1) = 0$. This is equivalent to

Theoreme 4.0.4. $\xi_1 \in \mathrm{Pic}(X_1)$ *is of Hodge type iff it can be lifted to a formal class* $\widehat{\xi} \in \varprojlim \mathrm{Pic}(X_n)_{\mathbb{Q}}$.

This is done by Berthelot-Ogus [4] using an idea of Deligne.

Step 2. The algebraisation, that is going from the formal class $\widehat{\xi}$ to a class in $\mathrm{Pic}(X)$. In this case we can make use of Grothendieck's formal existence theorem [1, Theo. 5.1.4], which gives an isomorphism

$$\mathrm{Pic}(X) \xrightarrow{\sim} \varprojlim \mathrm{Pic}(X_n).$$

And this then proofs the claim.

Remark 4.0.5. In the case of characteristic 0, Grothendieck that the absolute Hodge conjecture induces his variational one. In the case of characteristic $p > 0$, there is no p-adic analogue of the absolute Hodge conjecture, which would comprise the variational one. This makes its origin more mysterious than in the characteristic zero case.

5 General approach

As mentioned before, this case suggests to break up the problem in two steps – formal deformation and algebraisation.

$$\begin{tikzcd} K_0(X) \arrow[r] & \varprojlim K_0(X_n) \arrow[r] \arrow[l, bend left, squiggly, "\text{algebraisation}"] & K_0(X_1) \arrow[l, bend left, squiggly, "\text{deformation}"] \end{tikzcd}$$

Here we will study the deformation part. Unlike for the Picard group there seems to be no general approach to the algebraisation problem known. The main result reads:

Theoreme 5.0.6. *Let k be a perfect field of characteristic $p > 0$ and X/W a smooth projective scheme with closed fiber X_1, and $d = \dim X_1$. Suppose that $p > d + 6$. Then for $\xi_1 \in K_0(X_1)_{\mathbb{Q}}$ the following are equivalent*

1. *ξ is of Hodge type (i.e. $\Phi^{-1} \circ \mathrm{ch}(\xi_1) \in \oplus_r F^r \mathrm{H}^{2r}_{dR}(X/S)$).*
2. *There is $\widehat{\xi} \in \varprojlim K_0(X_n)_{\mathbb{Q}}$ such that $\widehat{\xi}\big|_{X_1} = \xi_1 \in K_0(X_1)_{\mathbb{Q}}$.*

Remark 5.0.7.
1. The case when the base ring is ramified over W is not treated here, as techniques related to integral p-adic Hodge theory are used that don't exist over a ramified base.
2. The condition $p > d + 6$ has technical reasons. Roughly, we need p to be big relative to d because of similar reasons of integrality as in the first remark.
3. In this paper, they lift ξ_1 directly to an element in $\varprojlim K_0(X_n)_{\mathbb{Q}}$ and not only its Chern character. So for algebraisation one might have to switch to a different pro-class with the same Chern character.

6 Idea of proof

Let $X_\bullet$ be the formal p-adic scheme associated to X. The following diagram displays the idea of the proof:

$$\begin{tikzcd} \oplus_{r\leqslant d} \mathrm{CH}^r_{\mathrm{cont}}(X_\bullet)_{\mathbb{Q}} \arrow[r] & \oplus_{r\leqslant d} \mathrm{CH}^r(X_1)_{\mathbb{Q}} \arrow[r, "\mathrm{Ob}"] & \oplus_{r\leqslant d} \mathrm{H}^{2r}_{\mathrm{cont}}(X_1, p(r)\Omega^{<r}) \\ K_0^{\mathrm{cont}}(X_\bullet)_{\mathbb{Q}} \arrow[u, "\Gamma"] \arrow[d, two heads] & & \\ \varprojlim K_0(X_n)_{\mathbb{Q}} \arrow[r] & K_0(X_1)_{\mathbb{Q}} \arrow[uu, hook, two heads] & \end{tikzcd} \tag{2}$$

where the top sequence is exact.

6.1 Construct the top sequence in the diagram

We do this for all r separately. We introduce the motivic pro-complex $\mathbb{Z}_{X_\bullet}(r) \in D_{pro}(X_1)_{\mathrm{Nis}}$ by glueiing the Suslin-Voevodsky complex to the syntomic complex of Fontaine-Messing over the logarithmic de Rham-Witt complex. This complex is part of a distinguished triangle which induces the desired exact sequence when we apply the functor of continuous cohomology.

6.2 Show that $\xi_1 \in K_0(X_1)_{\mathbb{Q}}$ is of Hodge type iff $\mathrm{Ob}(\xi_1) = 0$

Note that $K_0(X_1)_{\mathbb{Q}} = \oplus \mathrm{CH}^r(X_1)$. This is one of the main theorems of the paper. It follows from the connection homomorphism in the fundamental triangle and from the description of the crystalline cycle class maps in terms of continuous cohomology.

6.3 Construct the continuous Chern character Γ

For this we have to define continuous K-theory $K_0^{\text{cont}}(X_\bullet)$ and see that it maps surjectively

$$K_0^{cont}(X_\bullet) \to \varprojlim K_0(X_n)$$

so lifting classes from $K_0(X_1)$ to $\varprojlim K_0(X_n)$ amounts to the same as lifting to $K_0^{/}\text{cont}(X_\bullet)$. The construction of the Chern character

$$\text{ch} = \Gamma : K_0^{\text{cont}}(X_\bullet)_{\mathbb{Q}} \to \oplus_{r \leqslant d} \text{CH}^r_{\text{cont}}(X)_{\mathbb{Q}}$$

uses methods of Grothendieck and Gillet.

6.4 Show that this is an isomorphism

This uses results in topological cyclic homology theory due to Geisser-Hesselholt-Madsen.

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