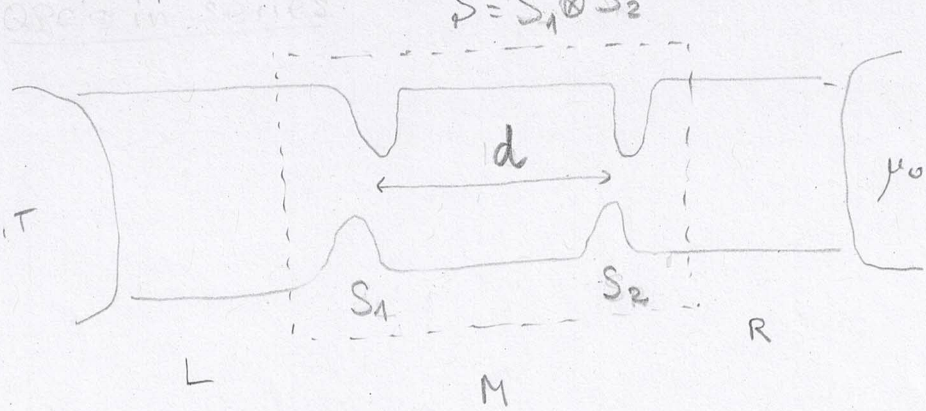
4.3 Multiple tunneling systems : QPC's in series

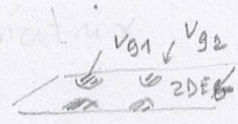
We consider now the situation where we have two tunneling barriers well inside the mesoscopic conductor. The situation where the tunneling barriers are at the interface with the reservoirs will be treated in a later chapter



Similar to the situation of the single QPC, the constrictions are well inside the mesoscopic system. Similar to before, if $\hat{V}_{ee}$ can be neglected, we can use the Landauer approach and the Ansatz (4.35) for the traveling states.

↳ the task is to obtain the transmission matrices of the combined system

e.g. laterally defined quantum dot



e.g. carbon nanotube with defects

New physical aspect: quantum interference of reflected and transmitted amplitudes at the barriers

Remember, in ch. 1 we have discussed the transmission T of incoherent scatterers in series and found for the case of two scatterers with transmissions probabilities T_1, T_2 that the total transmission probability was[(*)]

$$T_{1,2} = \frac{T_1 T_2}{1 - R_1 R_2} \quad (4.52), \quad \text{with} \quad T_1 = 1 - R_1, \; T_2 = 1 - R_2$$

How is this changing for coherent scatterers, i.e., if the phase information is kept?

Solution routes:

a) Evaluate the total S-matrix $S = S_1 \otimes S_2$

↑ to be defined soon

or

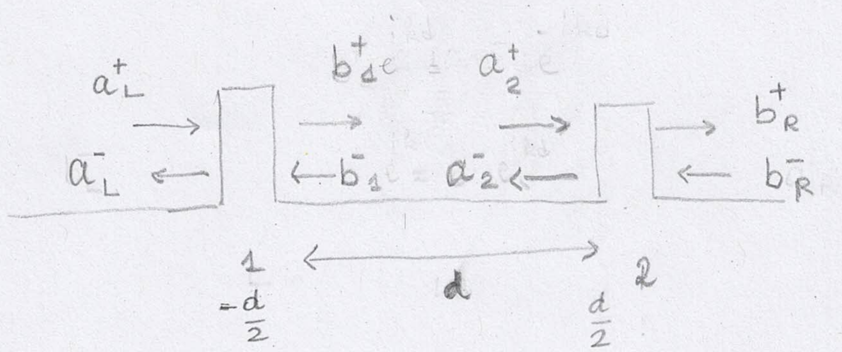
b) Evaluate the total transfer Matrix $M = M_1 M_2$

note that for the transfer matrix we just need an ordinary product of matrices

↳ exercise

(*) obtained from $T_1 T_2 \sum_{n=0}^{\infty} (R_1 R_2)^n \quad (= T_1 T_2 + T_1 R_2 R_1 T_2 + \cdots)$

Total S-matrix of two scatterers in series



$a_2^+ = b_1^+ e^{ikd}$

$b_1^- = a_2^- e^{-ik}$

$= a_2^- e^{ik}$

$$\vec{C}^{out} = S\vec{C}^{in}, \quad C^{in} = \begin{pmatrix} a^+_{L1} \\ a^+_{L2} \\ \vdots \\ b^-_{R1} \\ b^-_{R2} \\ \vdots \end{pmatrix}, \quad C^{out} = \begin{pmatrix} a^-_{L1} \\ a^-_{L2} \\ \vdots \\ b^+_{R1} \\ b^+_{R2} \\ \vdots \end{pmatrix}$$

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

Similarly for $S_1 = \begin{pmatrix} r_1 & t'_1 \\ t_1 & r'_1 \end{pmatrix}$, $S_2 = \begin{pmatrix} r_2 & t'_2 \\ t_2 & r'_2 \end{pmatrix}$

In general, $S \neq S_1 S_2$.

single channel case

$$\begin{pmatrix} a^-_L \\ b^+_R \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} a^+_L \\ b^-_R \end{pmatrix} \quad (4.53a) \text{ total S-matrix}$$

$$\begin{pmatrix} a^-_L \\ b^+_1 \end{pmatrix} = \begin{pmatrix} r_1 & t'_1 \\ t_1 & r'_1 \end{pmatrix} \begin{pmatrix} a^+_L \\ b^-_1 \end{pmatrix} \quad (4.53b) \text{ scattering at 1}$$

$$\begin{pmatrix} a^-_2 \\ b^+_R \end{pmatrix} = \begin{pmatrix} r_2 & t'_2 \\ t_2 & r'_2 \end{pmatrix} \begin{pmatrix} a^+_2 \\ b^-_R \end{pmatrix} \quad (4.53c) \text{ scattering at 2}$$

notice that incoming flux at 2 = out coming flux at 1

$$a_2^+ = b_1^+ e^{ikd}, \qquad b_1^- = a_2^- e^{ikd}$$

↳ We can eliminate a_2^+ and b_1^-; we find

from scattering 1

$$\begin{cases} \boxed{a_L^-} = r_1 a_L^+ + t_1' \underline{b_1^-} & (a) \\ b_1^+ = t_1 a_L^+ + r_1' b_1^- & (b) \end{cases}$$

from scattering 2

$$\begin{cases} a_2^- = r_2 a_2^+ + t_2' b_R^- & (c) \\ \boxed{b_R^+} = t_2 \underline{a_2^+} + r_2' b_R^- & (d) \end{cases}$$

↳ $\boxed{a_L^-} = r_1 a_L^+ + t_1' a_2^- e^{ikd}$

using (c) ⤳ $\boxed{a_L^-} = r_1 a_L^+ + t_1' e^{ikd} (r_2 a_2^+ + t_2' b_R^-)$ (e)

↳ one needs a_2^+ in terms of a_L^+ and b_R^-

from (b)

$$\begin{cases} a_2^+ = b_1^+ e^{ikd} \overset{(b)}{=} e^{ikd}\left(t_1 a_L^+ + r_1' b_1^-\right) \\ b_1^- = \left(a_2^- e^{ikd}\right) \underset{(c)}{=} e^{ikd}\left(r_2 a_2^+ + t_2' b_R^-\right) \end{cases}$$

↳ $$a_2^+ = e^{ikd} t_1 a_L^+ + e^{2ikd} r_1' r_2 a_2^+ + e^{2ikd} r_1' t_2' b_R^-$$

↳ $$a_2^+ \left(1 - e^{2ikd} r_1' r_2\right) = e^{ikd} t_1 a_L^+ + e^{2ikd} r_1' t_2' b_R^-$$

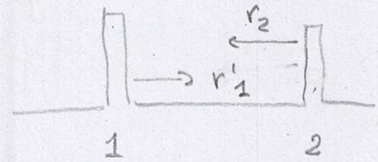
↳ $$\boxed{a_2^+ = \frac{e^{ikd} t_1}{1 - r_1' r_2 e^{2ikd}} a_L^+ + \frac{e^{2ikd} r_1' t_2'}{1 - r_1' r_2 e^{2ikd}} b_R^-} \quad (f)$$

Plugging back in (e) and observing that for the total S-matrix it holds

$$\begin{cases} a_L^- = r a_L^+ + t' b_R^- \\ b_R^+ = t a_L^+ + r' b_R^- \end{cases} \quad (g)$$

$$\Rightarrow \begin{cases} t = \dfrac{t_1 t_2 e^{ikd}}{1 - r'_1 r_2 e^{2ikd}} \\[2ex] t' = \dfrac{t'_1 t'_2 e^{ikd}}{1 - r'_1 r_2 e^{2ikd}} \\[2ex] r = r_1 + \dfrac{t'_1 t_1 e^{2ikd} r_2}{1 - r'_1 r_2 e^{2ikd}} \\[2ex] r' = r'_2 + \dfrac{t_2 t'_2 e^{2ikd} r'_1}{1 - r'_1 r_2 e^{2ikd}} \end{cases}$$

a round trip between the barriers involves $r_2 r'_1$



(4.54) a-d

↳ Transmission

$$T = |t|^2 = \frac{|t_1|^2 |t_2|^2}{|1 - |r'_1||r_2| e^{i(\varphi_1 + \varphi_2 + \varphi_D)}|^2}$$

where $r'_1 = |r'_1| e^{i\varphi_1}$

$r_2 = |r_2| e^{i\varphi_2}$

defining $T_1 = |t_1|^2 = |t'_1|^2$, $R_1 = 1 - T_1 = |r'_1|^2$

$T_2 = |t_2|^2 = |t'_2|^2$, $R_2 = 1 - T_2 = |r_2|^2$

we find

$$\boxed{T = \frac{T_1 T_2}{1 + R_1 R_2 - 2\sqrt{R_1}\sqrt{R_2}\cos(\underbrace{\varphi_1 + \varphi_2 + \varphi_D}_{\varphi_{tot}})}} \quad (4.55)$$

Note: $\varphi_{tot} = \varphi_1 + \varphi_2 + \varphi_D$ has usually two contributions.

For two scatterers at distance d it has

a) dynamical contribution φ_D

It comes from the propagating part e^{ikx} $\Rightarrow \varphi_D = 2kd$

b) static contribution $\varphi_S = \varphi_1 + \varphi_2$

It depends on the kind of scatterer

e.g. δ-potential $U(x) = U_0\,\delta(x) + U_0\,\delta(x-d)$

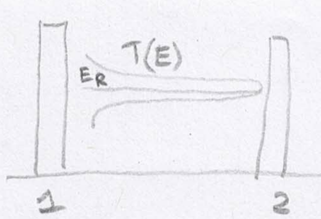
$$\Rightarrow \varphi_S = 2\tan^{-1}\frac{\hbar v}{U_0} \quad , \quad v = \frac{\hbar k}{m}$$

Resonant tunneling

Eq. (4.55) predicts resonances when $\varphi(E) = 2\pi n$, $n \in \mathbb{N}$

The general expression (4.55) simplifies in the case the transmission probability is a very sharp function of E.

This occurs when $T_1, T_2 \ll 1$, i.e. the tunneling probabilities out of the central region are small



$\hookrightarrow$ many reflections occur before a particle can escape

With $R_1 \approx 1$, $R_2 \approx 1$ and hence $T_1 << 1$, $T_2 << 1$, we first rewrite

$$T = \frac{T_1 T_2}{(1-\sqrt{R_1 R_2})^2 + 2\sqrt{R_1 R_2}(1-\cos\varphi)}$$

↳ to first order in T_1, T_2 (*)

$$T \approx \frac{T_1 T_2}{\left(\frac{T_1+T_2}{2}\right)^2 + 2\left(1-\left(\frac{T_1+T_2}{2}\right)\right)(1-\cos\varphi)}$$

Further, at resonance $\cos\varphi(E=E_R)=1$, i.e., $1-\cos\varphi(E=E_R)$ minimum

$$\hookrightarrow f(\varphi) = 1-\cos\varphi(E) \approx \underbrace{f(\varphi(E_R))}_{=0} + \underbrace{\frac{\partial f}{\partial\varphi}}_{=0}\frac{\partial\varphi}{\partial E}\Big|_{E=E_R}(E-E_R) + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial\varphi^2}}_{=1}\left(\frac{\partial\varphi}{\partial E}\right)^2(E-E_R)^2$$

↳ with $\boxed{\Gamma_{1,2} = T_{1,2}\left(\frac{d\varphi}{dE}\right)^{-1}}$ (4.56)

$$\boxed{T(E) \approx \frac{\Gamma_1\Gamma_2}{\left(\frac{\Gamma_1+\Gamma_2}{2}\right)^2 + (E-E_R)^2} = \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\frac{\Gamma}{(\Gamma/2)^2+(E-E_R)^2}}$$ (4.57)

with $\Gamma = \Gamma_1 + \Gamma_2$

$\Gamma_1 = \Gamma_2$

$\hookrightarrow T(E) = \frac{\Gamma_1^2}{\cancel{2\Gamma_1}} \frac{2\Gamma_1}{\Gamma_1^2 + (E-E_R)}$

⇒ effective broadening given by $\Gamma = \Gamma_1 + \Gamma_2$

It describes the finite lifetime of the resonant state E due to the coupling to the leads: $\tau = \hbar\Gamma^{-1}$

(*) $R_1 R_2 = (1-T_1)(1-T_2) = 1-(T_1+T_2) + T_1 T_2$

$\sqrt{R_1 R_2} = \sqrt{1-(T_1+T_2)+T_1T_2} \approx 1 - \frac{1}{2}(T_1+T_2) + O(T_1^2, T_2^2)$