Chapter 10. Lie groups and Lie algebras (an introduction)

10.1 Definition of a linear Lie group (from Ch. 3 of Cornwell Group Theory in Physics)

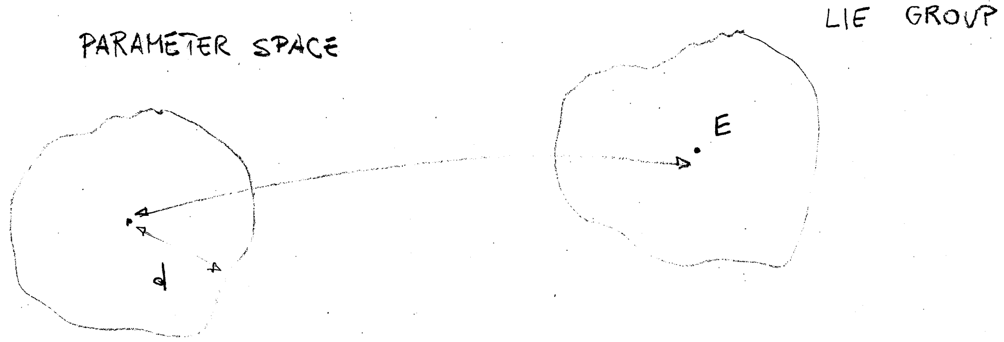
A Lie group embodies 3 different forms of mathematical structure

- a group: as we have introduced axiomatically in Chapter 1 and with its associated structures (classes of conjugation, cosets, subgroups, etc.)
- topological space
- analytic manifold.

Every Lie group important in physical problem is a LINEAR LIE GROUP which admits a compact axiomatic definition in terms of 4 properties.

The basic features of a Lie group:

- it contains a non-countable number of elements
- the elements lye "near" to the identity
- the structure of this region is determined by its corresponding real Lie algebra
- an analytical parametrization is required.



Def: Linear Lie group of dimension n (we need 4 properties to be satisfied

$\boxed{A}$ - $\mathcal{G}$ must posses at least one faithful finite-dimensional representation Γ.

If we suppose that Γ has dimension m, a distance is defined $d(T,T')$ between two elements $T, T' \in \mathcal{G}$.

$$d(T,T') = +\left\{ \sum_{j=1}^{m} \sum_{k=1}^{m} |\Gamma(T)_{jk} - \Gamma(T')_{jk}|^2 \right\}^{1/2}$$

$d: \Gamma \longrightarrow \mathbb{R}$ fulfill the following distance properties:

(i) $d(T',T) = d(T,T')$

(ii) $d(T,T) = 0$

(iii) $d(T,T') > 0$ if $T \neq T'$

(iv) if T, T' and $T'' \in \mathcal{G} \Rightarrow d(T,T'') \leq d(T,T') + d(T',T'')$

The metric is inherited from the standard one of $\mathbb{C}^{m^2}$ thus all the properties (i)-(iv) follow.

We denote with M_δ, a sphere of radius δ centered on the identity E, the elements $T \in \mathcal{G}$ such that $d(T,E) < \delta$.

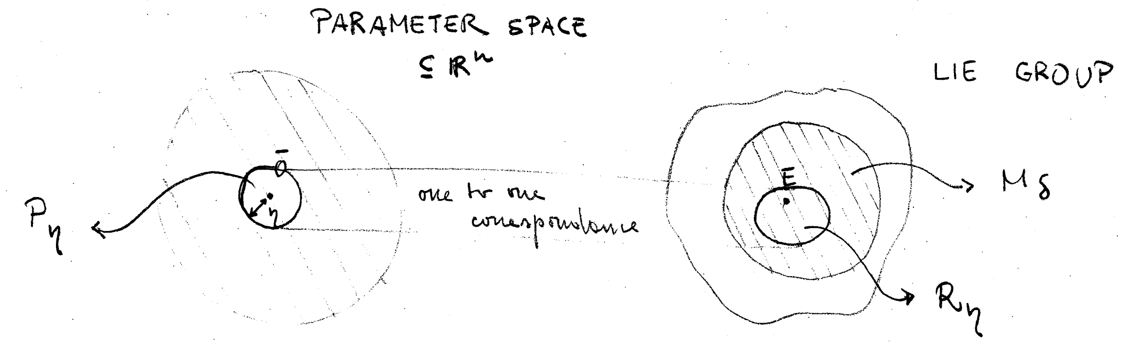
$\boxed{B}$ There must exist a $\delta > 0$ such that every element $T \in \mathcal{G}$ lying in the sphere M_δ of radius δ centered on the identity can be parametrized by n real parameters $x_1, x_2, \dots, x_n$ (no two sets of parameters corr. to the same $T \in \mathcal{G}$), the identity E being parametrized by $x_1 = x_2 = \dots = x_n = 0$,

$\boxed{C}$ There must exist a $\eta > 0$ such that every point in $\mathbb{R}^n$ for which

$$\left\{ \sum_{j=1}^{n} x_j^2 \right\}^{1/2} < \eta \qquad (*)$$

correspond to some element $T \in M_\delta$. We call R_η such region.

$R_\eta \subseteq M_\delta$ and there must be a one-to-one correspondence between the elements in R_η and the parameters satisfying (*).



Thanks to the last conditions the group multiplication operation is expressible in terms of analytic functions. $T(x_1, x_2, \dots, x_n) \in R_\eta$ and $\Gamma(x_1, x_2, \dots, x_n) \equiv \Gamma(T(x_1, x_2, \dots, x_n)) \quad \forall \; T \in R_\eta$.

$\boxed{D}$ Each of the matrix elements of $\Gamma(x_1, x_2, \dots, x_n)$ must be an analytic function of $x_1, x_2, \dots, x_n$ $\forall \; (x_1, x_2, \dots x_n)$ satisfying (*).

Analytic means that all functions $\Gamma_{jk}(\{x_h\})$ can be expanded in power series in $(x - x_0)_k$ $\Rightarrow$ all derivatives are well defined in every point satisfying (*).

Conditions $\boxed{A}$ - $\boxed{D}$ imply the very important <u>theorem: I</u>

The matrices $a_1, \dots, a_n$ defined as:

$$(a_p)_{jk} = \left(\frac{\partial \Gamma_{jk}}{\partial x_p} \right)_{x_1 = x_2 = \dots = x_n = 0}$$

form the basis for an n-dimensional real vector space.

<u>proof</u> It is required to show that the only solution of the equation $\sum_{p=1}^{n} \lambda_p a_p = 0$ with $\{\lambda_p\} \in \mathbb{R}^n$ is $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$. Let $\Gamma_{jk} = A_{jk} + iB_{jk}$

and $\mathcal{S}' = \{\{A_{jk}\}, \{B_{jk}\}\}$

There must be $\mathcal{S} = \{C_p\} \quad p = 1, \dots, n \quad \mathcal{S} \subset \mathcal{S}'$

Such that all elements in $\mathcal{S}'$ are analytic functions of elements of $\mathcal{S}$ (there are only n variables $\{x_1, \dots x_n\}$!). Since the mapping between the elements in P_η and the ones in R_η is one to one $\det\left(\frac{\partial c_j}{\partial x_p}\right)_{x_1 = x_2 \dots} \neq 0$ $\Rightarrow$ the only solution of $\sum_p \lambda_p \left(\frac{\partial c_j}{\partial x_p}\right)_{\bar{x}=0}$ is $\bar{\lambda} = 0$. Consequently for the larger set $\sum_{p=1}^{n} \lambda_p \left(\frac{\partial \Gamma_{jk}}{\partial x_p}\right)_{\bar{x}=0} = 0$. ∎

Notice that $a_1, \dots, a_n$ form the basis for a real vector space but they are NOT, in general, real matrices. The matrices $a_1, \dots a_n$ form the basis of a real Lie algebra. (see later)

The multiplication and the inversion operation are both "analytic".

$U \subset R_\eta$: if $T, T' \in U \Rightarrow T'' = TT' \in R_\eta$. We associate now:

$$T \longleftrightarrow x_1, \dots, x_n$$

$$T' \longleftrightarrow x_1', \dots, x_n' \qquad x_j'' = f_j(x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n') \quad j = 1, 2, \dots, n$$

$$T'' \longleftrightarrow x_1'', \dots, x_n''$$

As $TE = T \in T \in R_\eta$ and similarly $ET' = T'$, it follows.

$$x_j = f_j(x_1, \dots, x_n, 0, 0 \dots, 0)$$

$$x_j' = f_j(0, \dots, 0, x_1', \dots, x_n')$$

Theorem II: The functions $f_j(x_1, \dots, x_n, x_1', \dots, x_n')$ are analytic functions $\forall\, T, T' \in U$.

Theorem III The coordinates $(x_1', x_2', \dots x_n')$ of T^{-1} are given in terms of the coordinates $(x_1, x_2, \dots x_n)$ of T by:

$$x_j' = g_j(x_1, x_2, \dots, x_n) \qquad j = 1, 2, \dots, n$$

where $g_1, g_2, \dots, g_n$ are analytic functions, and T and T^{-1} are both in R_η.

proof of Theorem II

with the notation of the Theorem I, as the Jacobian $\det\left(\frac{\partial c_j}{\partial x_p}\right)_{\{x_p\}=0}$ is non-zero and as the mapping from $(x_1 \dots x_n)$ to $(c_1 \dots c_n)$ is bijective, the inverse function theorem implies that we can write $x_j = \phi_j(c_1, c_2, \dots, c_n)$ where the functions ϕ_j are analytic functions of $c_1, \dots, c_n$.

In particular for the product $T'' = TT'$, $x_j'' = \phi_j(c_1'', c_2'', \dots, c_n'')$ where c_j'' are directly related to the real and imaginary parts of the elements of $\Gamma(T'')$. Thus they are analytic functions of the real and imaginary parts of $\Gamma(T)$ and $\Gamma(T')$ (as $\Gamma(T'') = \Gamma(T)\Gamma(T')$) and hence, by $\boxed{A}$ of $x_1, \dots, x_n$ and $x_1', \dots x_n'$.

proof of Theorem III

By the equation

$$x''_j = f_j(x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n')$$

Since $T'' = E$ has coordinate $(0, 0, \dots, 0)$ we obtain

$$0 = f_j(x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n') \qquad j = 1, \dots, n.$$

The state result follows immediately from the implicit function theorem.

Notice:

1) The definition just given requires a parametrization only of the group elements belonging to a small neighborhood of E.

2) In some cases a simple set of parameters $x_1, \dots x_n$ parametrizes a large part or even the whole group. It is NOT required, though.

Example I: The multiplicative group of real numbers

$\mathcal{G}: \mathbb{R}\setminus\{0\}$. $t_3 = t_1 t_2$ $\quad E = 1$. $\quad t^{-1} = \frac{1}{t}$.

$\Gamma(t) = [t]$ which is a representation of dimension 1. The metric

$d(t,t') = |t-t'|$. $\Rightarrow$ $d(t,E) = |t-1|$

$\delta = \frac{1}{2}$ $\Rightarrow$ $M_{1/2} = \{t \mid \frac{1}{2} < t < \frac{3}{2}\}$

Representation $t = \exp x_1$ $\quad x_1 = 0 \iff t = 1 \longleftrightarrow E$

If we now take $\eta = \log\frac{3}{2}$ $\quad -\log\frac{3}{2} < x_1 < \log\frac{3}{2}$ $\quad \Rightarrow \frac{1}{2} < \frac{2}{3} < \exp(x_1) < \frac{3}{2}$

$\Gamma(x_1)_{11} = \exp(x_1)$ analytic. $\quad a_1 = \left.\frac{\partial \Gamma_{11}}{\partial x_1}\right|_{x_1=0} = 1$. The parametrization $t = \exp x_1$ extends to all $t > 0$, a subgroup of $\mathcal{G}$. Every group element with $t < 0$ can be written in the form $t = (-1)\exp x_1$.

Example II The groups $O(2)$ and $SO(2)$

$O(2)$ is the group of all real orthogonal 2×2 matrices A, $SO(2)$ being the subgroup for which $\det A = 1$.

If $A \in O(2)$ $\quad \Gamma(A) = A$ is a faithful finite-dimensional representation.
The orthogonality condition requires $AA^T = 1 = A^TA$ in other terms

$$\sum_k A_{ik}(A^T)_{kj} = \delta_{ij} = \sum_{k=1}^{2} A_{ik}A_{jk}$$
$$\sum_k (A^T)_{ik}A_{kj} = \delta_{ij} = \sum_{k=1}^{2} A_{ki}A_{kj}$$

$$\begin{cases} A_{11}^2 + A_{12}^2 = A_{22}^2 + A_{12}^2 = A_{11}^2 + A_{21}^2 = A_{22}^2 + A_{21}^2 = 1 \\ A_{11}A_{21} + A_{22}A_{12} = A_{11}A_{12} + A_{22}A_{21} = 0 \end{cases}$$

We deduce immediately $A_{11}^2 = A_{22}^2$ and $A_{12}^2 = A_{21}^2$. There are thus 2 sets of solutions:

(i) $A_{11} = A_{22}$ and $A_{21} = -A_{12}$ $\Rightarrow$ $\det A = A_{11}A_{22} - A_{12}A_{21} = A_{11}^2 + A_{21}^2 = 1$

(ii) $A_{11} = -A_{22}$ and $A_{21} = A_{12}$ $\Rightarrow$ $\det A = -A_{11}^2 - A_{21}^2 = -1$

The distance to the identity: $d(A,1)$

(i) $$\left[(A_{11}-1)^2 + (A_{22}-1)^2 + A_{12}^2 + A_{21}^2\right]^{1/2} =$$

$$= \left[2(A_{11}^2 + A_{12}^2) + 2 - 4A_{11}\right]^{1/2} = 2(1 - A_{11})^{1/2}$$

(ii) $$\left[A_{11}^2 + 1 - \cancel{2A_{11}} + A_{22}^2 + 1 - \cancel{2A_{22}} + A_{12}^2 + A_{21}^2\right]^{1/2} = 2$$

We can choose $\delta = \sqrt{2}$ $\Rightarrow$ $\boxed{B}$ requires the parametrization of (i) but not of (ii).

$$A = \Gamma(A) = \begin{vmatrix} \cos x_1 & \sin x_1 \\ -\sin x_1 & \cos x_1 \end{vmatrix}$$

$x_1 = 0$ corresponds to the group identity and the parameter space dimension $n = 1$. if $-\frac{\pi}{3} < x_1 < \frac{\pi}{3}$ $\Rightarrow$ $A(x_1) \in M_\delta$ $\Rightarrow$ $\boxed{C}$ is satisfied. The parametrization above with $-\pi \leq x_1 \leq \pi \Leftrightarrow SO(2)$ condition $\boxed{D}$ is satisfied $\Rightarrow$ Both $SO(2)$ and $O(2)$ are linear Lie groups of dimension 1.

$$a_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The set ii) is obtained for example by $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos x_1 & \sin x_1 \\ -\sin x_1 & \cos x_1 \end{bmatrix}$

$$= \begin{vmatrix} -\sin x_1 & \cos x_1 \\ \cos x_1 & \sin x_1 \end{vmatrix}.$$

Example III The group SU(2) it is the group of the unitary matrices u with $\det u = 1$. A faithful finite-dimensional representation is provided by $\Gamma(u) = u$.

$$u u^+ = u^+ u = 1_2 \quad \Rightarrow \quad \sum_k u_{ik} u_{jk}^* = \delta_{ij}$$
$$\sum_k u_{ki}^* u_{kj} = \delta_{ij}$$

$$|u_{11}|^2 + |u_{12}|^2 = |u_{22}|^2 + |u_{12}|^2 = |u_{11}|^2 + |u_{21}|^2 = |u_{22}|^2 + |u_{21}|^2 = 1$$

$$\Rightarrow \quad |u_{11}|^2 = |u_{22}|^2 \quad \text{and} \quad |u_{12}|^2 = |u_{21}|^2 \quad \Rightarrow \quad \begin{array}{ll} u_{11} = a e^{i\varphi_1} & u_{22} = a e^{i\varphi_2} \\ u_{12} = b e^{i\varphi_3} & u_{21} = b e^{+i\varphi_4} \end{array}$$

$$u_{11} u_{21}^* + u_{12} u_{22}^* = u_{11}^* u_{12} + u_{21}^* u_{22} = 0$$

$$\left. \begin{array}{l} u_{11} u_{21}^* = -u_{12} u_{22}^* \\ u_{11}^* u_{12} = -u_{22} u_{21}^* \end{array} \right\} \Rightarrow \quad \varphi_1 - \varphi_4 = \varphi_3 - \varphi_2 + (2n+1)\pi$$

Moreover $\quad a^2 e^{i(\varphi_1+\varphi_2)} - b^2 e^{i(\varphi_3+\varphi_4)} = 1$

$$\Rightarrow \quad a^2 \sin(\varphi_1 + \varphi_2) - b^2 \sin(\varphi_3 + \varphi_4) = 0 \quad \text{but} \quad \varphi_1 + \varphi_2 = \varphi_3 + \varphi_4 + (2n+1)\pi$$

$$\Rightarrow \quad (a^2 + b^2) \sin(\varphi_1 + \varphi_2) = 0 \quad \text{since} \quad a^2 + b^2 \neq 0 \quad \varphi_1 = -\varphi_2 \quad \Rightarrow \quad \varphi_3 = \varphi_4 + (2n+1)\pi$$

$$u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = a^2 + b^2 = 1.$$

If we write $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ we obtain the condition $\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 = 1$.

and thus the parametrization

$$\alpha_1 = \sqrt{1 - (x_1^2 + x_2^2 + x_3^2)} \qquad \alpha_2 = x_3 \qquad \beta_1 = x_2 \qquad \beta_2 = x_1$$

for $x_1 = x_2 = x_3 = 0$ we obtain the identity. and:

$$d(u, \mathbb{1}) = \left[2|\alpha_1 + i\alpha_2 - 1|^2 + 2|\beta_1 + i\beta_2|^2 \right]^{1/2} =$$

$$= \left[2(\alpha_1 - 1)^2 + 2\alpha_2^2 + 2\beta_1^2 + 2\beta_2^2 \right]^{1/2} =$$

$$= (4 - 4\alpha_1)^{1/2} = 2\left\{ 1 - \left[1 - (x_1^2 + x_2^2 + x_3^2) \right]^{1/2} \right\}^{1/2}$$

thus $d(u, \mathbb{1}) < \delta \iff 1 - \left[1 - (x_1^2 + x_2^2 + x_3^2)\right]^{1/2} < \frac{\delta^2}{4}$ or in other terms $\left[1 - (x_1^2 + x_2^2 + x_3^2)\right] > \left(1 - \frac{\delta^2}{4}\right)^2$ $\quad x_1^2 + x_2^2 + x_3^2 < 1 - \left(1 - \frac{\delta^2}{4}\right)^2 =$

$$= \frac{\delta^2}{2} - \frac{\delta^4}{16} = \frac{1}{4}\left(2\delta^2 - \frac{1}{4}\delta^4\right)$$

$\Rightarrow$ if $\delta < 2\sqrt{2}$ and $\eta = \frac{1}{2}\left(2\delta^2 - \frac{1}{4}\delta^4\right)^{1/2}$ conditions $\boxed{B}$ and $\boxed{C}$ are satisfied and M_δ and R_η coincide. Condition $\boxed{D}$ is clearly true, so $SU(2)$ is a linear Lie group of dimension 3. The vectors e_p are:

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

10.2 The connected components of a linear Lie group

Definition Connected component of a Linear Lie group $\mathcal{G}$. A maximal set of elements $T \in \mathcal{G}$ that can be obtained from each other by continuously varying one or more of the matrix elements $\Gamma(T)_{jk}$ of the faithful finite dimensional representation Γ.

Example I The multiplicative group of real numbers.

$t > 0$ is a connected component

$t < 0$ is a connected component

$t = 0$ is excluded and separates the 2 connected components

Example II The groups $O(2)$ and $SO(2)$

$SO(2)$ is parametrized as $\begin{pmatrix} \cos x_1 & \sin x_1 \\ -\sin x_1 & \cos x_1 \end{pmatrix}$ $\quad -\pi \leq x_1 \leq \pi$

The set (ii) of page 125 correspond to the parametrization

$$\begin{pmatrix} \sin x_1 & \cos x_1 \\ \cos x_1 & -\sin x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos x_1 & \sin x_1 \\ -\sin x_1 & \cos x_1 \end{pmatrix}$$

The set (i) $\equiv SO(2)$ forms a connected component of $O(2)$. So does set (ii). Notice though that set (ii) is NOT a subgroup (e.g. it lacks the identity). In a connected component $\det \Gamma(T)$ should vary continuously. It is instead 1 for $SO(2)$ and -1 for the set (ii).

Theorem I The connected component of a linear Lie group $\mathcal{G}$ that contains the identity E is an invariant subgroup of $\mathcal{G}$. This component is often referred to as "the connected subgroup of $\mathcal{G}$".

proof. Let $\mathcal{S}$ be the connected component of $\mathcal{G}$ that contains E

- $\mathcal{S}$ is a subgroup: $S \in \mathcal{S} \Rightarrow S'S^{-1}$ with $S' \in \mathcal{S}$ in a connected component of $\mathcal{S}$. But with $S' = S$ this component contains $E \Rightarrow$ it is $\mathcal{S}$ itself. $\Rightarrow \forall S', S \in \mathcal{S}$ $S'S^{-1} \in \mathcal{S} \Rightarrow \mathcal{S}$ is a subgroup. (contain E, S^{-1} + closure, associativity is inherited from $\mathcal{G}$).
- invariant subgroup: $X \in \mathcal{G}$ XSX^{-1} when $S \in \mathcal{S}$. XSX^{-1} in a connected component and it contains E ($S = E \Rightarrow XSX^{-1} = E$). $\Rightarrow XSX^{-1} \in \mathcal{S}$. ∎

Definition Connected linear Lie group.

A linear Lie group is said to be "connected" if it possesses only one connected component.

Notice The parametrization $y_1 \dots y_n$ that allows a Lie group to be called connected requires continuity of $\Gamma_{jk}(T)$, not analiticity. The sets $x_1 \dots x_n$ of the def of a Lie group concern local properties in the vicinity of the identity.

In general $x_1, \dots x_n$ and $y_1, \dots, y_n$ are NOT interchangeable.

Example III The group $SU(2)$

We have already proven that $u \in SU(2)$ implies

$$u = \begin{vmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{vmatrix} \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1$$

Using the modulus phase parametrization of complex numbers

$$\alpha = a e^{iy_2} \qquad \beta = b e^{iy_3} \qquad \text{with} \quad a^2 + b^2 = 1$$

$$\Rightarrow \qquad \alpha = \cos y_1 \, e^{iy_2} \quad \beta = \sin y_1 \, e^{iy_3}$$

$$0 \leq y_1 \leq \frac{\pi}{2} \qquad 0 \leq y_2 \leq 2\pi \qquad 0 \leq y_3 \leq 2\pi$$

$$\Rightarrow \qquad u = \Gamma(u) = \begin{bmatrix} \cos y_1 \, e^{iy_2} & \sin y_1 \, e^{iy_3} \\ -\sin y_1 \, e^{-iy_3} & \cos y_1 \, e^{-iy_2} \end{bmatrix}$$

Since $\Gamma_{jk}(y_1, y_2, y_3)$ are continuous functions $\Rightarrow$ $SU(2)$ is a connected linear Lie group. The parametrization y_1, y_2, y_3 does not provide, though a bijective relation to the group matrices in the vicinity of E since

$$y_1 = 0 \quad y_2 = 0 \quad 0 \leq y_3 \leq 2\pi \quad \longleftrightarrow \quad E.$$

Example IV The groups $O(3)$ and $SO(3)$

$O(3)$ is isomorphic to the group of all rotations in $\mathbb{R}^3$ and $SO(3)$ is isomorphic to the subgroup of proper rotations. The group of all rotations has 2 connected components of which $SO(3)$ contains the identity. The other connected component is obtained by multiplying each element of $SO(3)$ by the inversion matrix $I = -\mathbb{1}_3$.

Definition Compact linear Lie group of dimension n.

A linear Lie group of dimension n with a finite number of connected components is compact if the parameters $y_1, y_2, \dots, y_n$ range over the closed finite intervals $a_j \leq y_j \leq b_j$ $j = 1, 2, \dots, n$.

The importance of the distinction between compact and non-compact groups lies in the fact that the representation theory of compact Lie groups is very largely the same as that for finite groups.

10.3 The role of Lie algebras (from Ch. 10 of Cornwell 'Group Theory in Physics')

Most of the characteristics of Lie groups are associated to their "local" properties in the vicinity of the identity. The matrices a_p introduced at page 122 and the vector space generated by them will play a central role.

10.3.1 One-parameter subgroups

The first concept to be introduced is the one of:

Definition One-parameter subgroup of a linear Lie group: it is a Lie subgroup of $\mathcal{G}$ consisting of elements $T(t)$ which depend on a real parameter $t \in (-\infty, +\infty)$ such that

$$T(s)\,T(t) = T(s+t)$$

In particular, if $\mathcal{G}$ is a group of $m \times m$ matrices $\Rightarrow$ we have a one parameter subgroup of matrices with matrix product.

Clearly $T(s)T(t) = T(t)T(s)$ => a one-parameter Lie subgroup is an Abelian group. Moreover $T(s=0) = E$ since $T(s=0)T(t) = T(t)$. Obviously this subgroup is a Lie group of dimension 1 => $dA/dt|_{t=0} \neq 0$

<u>Theorem</u> Every one-parameter subgroup of a linear Lie group $\mathcal{G}$ of $m\times m$ matrices is formed by exponentiation of $m\times m$ matrices.

$$A(t) = \exp\{ta\}$$

and $$a = \frac{dA}{dt}\Big|_{t=0}$$

<u>proof</u> Let $B(t) = A(t)\exp\{-t\dot{A}(0)\}$ all well defined since $A(t)$ is a 1D linear Lie group.

$$\dot{B}(t) = \{\dot{A}(t) - A(t)\dot{A}(0)\}\exp\{-t\dot{A}(0)\}$$

But $\dot{A}(t) = \lim_{s\to 0}\frac{1}{s}\{A(t+s) - A(t)\} = \lim_{s\to 0} A(t)\{A(s) - A(0)\}/s = A(t)\dot{A}(0)$

=> $\dot{B}(t) = 0 \quad \forall t$ => $B(t) = B(0) = \mathbb{1} \quad \forall t \in \mathbb{R}$.

$$\mathbb{1} = A(t)\exp\{-t\dot{A}(0)\}$$

=> by multiplying both sides on the right by $\exp\{t\dot{A}(0)\}$

$$A(t) = \exp\{t\dot{A}(0)\} = \exp\{ta\} \qquad \blacksquare$$

We make now the following observations: Any proper rotation is a rotation of θ_0 around a certain axis => the set of all rotations around a certain axis form a one-parameter subgroup. => every matrix of $SO(3)$ must lie in some one parameter subgroup.

$$A(t) = \exp\{ta\} \qquad A, a \in GL(3,\mathbb{R})$$

$$A(t)^{-1} = \exp\{-ta\} \overset{A\in SO(3)}{=} A^T(t) = \exp\{t\, a^T\}$$

$\Rightarrow$ a is ANTISYMMETRIC. Thus every element of $SO(3)$ is obtained by exponentiation from some 3×3 real antisymmetric matrix.

The set of all real antisymmetric matrices forms a three dimensional real vector space. (prove by exercise). We give the "convenient" basis:

$$a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice: The commutator $[a,b]$ of real antisymmetric matrices is also a real antisymmetric matrix

$$[a,b]^T = [ab - ba]^T = [b^Ta^T - a^Tb^T] = [ba - ab] = -[a,b].$$

$\Rightarrow$ is still a member of the same vector space. This motivates the

<u>Definition</u> Real Lie algebra $\mathcal{L}$

A real Lie algebra $\mathcal{L}$ of dimension $n\,(\geq 1)$ is a real vector space of dimension n equipped with a Lie product $[a,b]$ defined for every $a, b \in \mathcal{L}$ such that:

(i) $[a,b] \in \mathcal{L} \quad \forall\, a, b \in \mathcal{L}$

(ii) $[\alpha a + \beta b, c] = \alpha[a,c] + \beta[b,c] \quad \forall\, a,b,c \in \mathcal{L}$ and $\alpha, \beta \in \mathbb{R}$

(iii) $[a,b] = -[b,a] \quad \forall\, a,b \in \mathcal{L}$

(iv) $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ (Jacobi's identity)

For a Lie algebra of matrices the commutator $[a,b]$ will be always <u>defined</u> by

$$[a,b] = ab - ba$$

and the conditions (ii), (iii) and (iv) are automatically satisfied. For Lie algebras of linear operators the commutator $[a,b]$ is <u>defined</u> by

$$[a,b]\phi = a(b\phi) - b(a\phi)$$

Let $a_1, \dots, a_n$ be a basis of the real vector space of $\mathcal{L}$. As $[a_p, a_q] \in \mathcal{L}$ $\forall\, p,q = 1,2,\dots,n$ there exists a set of n^3 real numbers c_{pq}^r known as "structure constants of $\mathcal{L}$ with respect to the basis $a_1 \dots a_n$ defined by:

$$[a_p, a_q] = \sum_{r=1}^{n} c_{pq}^r a_r$$

Notice that c_{pq}^r are NOT independent. conditions (iii) and (iv) of the definition of $\mathcal{L}$ imply: $c_{pq}^r = -c_{qp}^r$ and $\sum_s \left(c_{pq}^s c_{rs}^t + c_{qr}^s c_{ps}^t + c_{rp}^s c_{qs}^t \right) = 0$

Moreover any commutator can be expressed in terms of the structure constants:

$$\left.\begin{array}{l} a = \sum_p \alpha_p a_p \\ b = \sum_q \beta_q a_q \end{array}\right. \Rightarrow [a,b] = \sum_{p,q,r=1}^{n} \alpha_p \beta_q c_{pq}^r a_r$$

In particular, in the real Lie algebra $\mathcal{L} = so(3)$ associated with $SO(3)$ with the basis elements a_1, a_2, a_3

$$[a_1, a_2] = -a_3 \quad [a_2, a_3] = -a_1 \quad [a_3, a_1] = -a_2$$

in other terms $c_{ij}^k = -\varepsilon_{ijk}$ where ε_{ijk} is the Levi-Civita symbol.

Proof of the relation

$$\sum_s \left(C_{pq}^{s} C_{rs}^{t} + C_{qr}^{s} C_{ps}^{t} + C_{rp}^{s} C_{qs}^{t} \right) = 0$$

The starting point is the jacoby identity written for the basis matrices e_p, e_q, e_r.

$$[a_r, [e_p, e_q]] + [e_p, [e_q, e_r]] + [e_q, [e_r, e_p]] = 0$$

We rewrite the first double commutator in terms of the structure constants.

$$[a_r, [e_p, e_q]] = \sum_s C_{pq}^{s} [a_r, e_s] = \sum_{st} C_{pq}^{s} C_{rs}^{t} a_t$$

Analogously we can proceed with the other two double commutators.

$$\sum_{st} \left(C_{pq}^{s} C_{rs}^{t} + C_{qr}^{s} C_{ps}^{t} + C_{rp}^{s} C_{qs}^{t} \right) a_t = 0$$

Since a_t is a basis of a vector space we obtain the final relation.

The commutation relations just introduced take a very familiar form when the real Lie algebra associated with the group of linear operators $\hat{T}$ corresponding to the group of proper rotations in $\mathbb{R}^3$ is considered.

$$T = \exp(ta)$$

It follows for the operator $\hat{T}$

$$\hat{T} f(r) = f(\{\exp(ta)\}^{-1} r) = f(\{\exp(-ta)\} r)$$

Since $\quad a = \lim_{t\to 0} \dfrac{\exp(ta) - 1}{t} \quad$ it is natural to define the operator $\hat{a}$:

$$\hat{a} f(r) = \lim_{t\to 0} \left[f(\{\exp(-ta)\} r) - f(r) \right] / t$$

But $\quad f(\{\exp(-ta)\} r) \simeq f(\{1 - ta\} r) \simeq f(r) - t(ar)^T \operatorname{grad} f(r)$

Consequently

$$\hat{a} f(r) = - r^T a^T \operatorname{grad} f(r)$$

$$\Rightarrow \quad \hat{a}_1 = -(x\ y\ z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} = + y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}$$

$$\hat{a}_2 = -(x\ y\ z) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} = - x \frac{\partial}{\partial z} + z \frac{\partial}{\partial x}$$

$$\hat{a}_3 = -(x\ y\ z) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$

Exatly the same equations written for the matrices a_1, a_2, a_3 are valid for the corresponding operators:

$$[\hat{a}_1, \hat{a}_2] = -\hat{a}_3$$
$$[\hat{a}_2, \hat{a}_3] = -\hat{a}_1$$
$$[\hat{a}_3, \hat{a}_1] = -\hat{a}_2$$

Equations above are directly connected to the quantum mechan. angular momentum operators:

$$\hat{L}_x = -i\hbar\,\hat{a}_1 \qquad \hat{L}_y = -i\hbar\,\hat{a}_2 \qquad \hat{L}_z = -i\hbar\,\hat{a}_3$$

It follows that

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

There is an intimate connection between the quantum theory of angular momentum and the group of proper rotations in $\mathbb{R}^3$.

The determination of the basis functions of the irreducible representati of the group of proper rotations in $\mathbb{R}^3$ can be reduced to the construction of the simultaneous eigenfunctions of L_z and $L^2 = L_x^2 + L_y^2 + L_z^2$. The method makes use of the ladder operators $L_\pm \equiv L_x \pm i L_y$ which alone justify the extension of the concept of real Lie algebra.

<u>Definition</u> Complex Lie algebra $\mathcal{L}$.

A complex Lie algebra $\mathcal{L}$ of dimension n (≥ 1) is a complex vector space of dimension n equipped with a "Lie product" or "commutator" possessing the properties (i)-(iv) listed in the definition of real Lie algebra. In (ii) α and β are complex numbers.

Real Lie algebras for general linear Lie groups

For $\mathcal{G} = SO(3)$ the existence of a real Lie algebra stemmed naturally from the occurrence of one-parameter subgroups. In the general case one takes the opposite route.

(a) Existence of a real Lie algebra $\mathcal{L}$ for every linear Lie group $\mathcal{G}$

Notice: at this point we identify the Lie group with its faithful m dimensional representation. We consider thus only groups of matrices.

<u>Def</u>: Analytic curve in $\mathcal{G}$

Let $x_1(t), \dots, x_n(t)$ be a set of real analytic functions of t defined in some interval $[0, t_0]$ where $t_0 > 0$, such that $x_j(0) = 0$ $\forall j = 1, \dots, n$. and the point $x_1(t) \dots x_n(t)$ satisfies $\sum_j x_j^2(t) < \eta^2$ $\forall$ $t \in [0, t_0]$. $\Rightarrow$ The set of $m \times m$ matrices $A(t) \in \mathcal{G}$ defined by $A(t) = A(x_1(t) \dots x_n(t))$ are said to form an "analytic curve" in $\mathcal{G}$.

$A(0) = \mathbb{1}_m$ $\Rightarrow$ every analytic curve starts from the identity. There is no requirement though that an analytic curve is a one-param. subgroup of $\mathcal{G}$.

<u>Def</u>: tangent vector of an analytic curve in $\mathcal{G}$: $a = \left.\dfrac{dA(t)}{dt}\right|_{t=0}$.

Theorem I The tangent vector of <u>any</u> analytic curve in $\mathcal{G}$ is a member of the of the vector space V spanned by the matrices $e_j = \dfrac{\partial A}{\partial x_j}$. Conversely any member of V is the tangent vector of some analytic curve in $\mathcal{G}$.

proof

- $$\frac{dA(t)}{dt} = \sum_{p=1}^{n} \frac{\partial A}{\partial x_p} \frac{dx_p}{dt} \Rightarrow a = \sum_{p=1}^{n} \dot{x}_p(0)\, a_p \Rightarrow a \in V$$

- Conversely, suppose: $a = \sum_{p=1}^{n} \lambda_p a_p$ is any member of $V \Rightarrow x_j(t) = \lambda_j t$ defines a proper analytic curve.

Theorem II: If a and b are the tangent vectors of the analytic curves $A(t)$ and $B(t)$ in $\mathcal{G}$ $\Rightarrow$ $[a,b]$ $(= ab - ba)$ is the tangent vector of the analytic curve

$$C(t) = A(\sqrt{t})\, B(\sqrt{t})\, A^{-1}(\sqrt{t})\, B^{-1}(\sqrt{t}).$$

$C(t)$ is an analytic curve thanks to the theorem on composition of elements in $\mathcal{G}$. (to be verified)

$s = \sqrt{t}$

$$A(s) = 1 + sa + \frac{1}{2} s^2 a' + o(s^2) \text{ where } a' = \left.\frac{d^2 A(t)}{dt^2}\right|_{t=0}$$

$$B(s) = 1 + sb + \frac{1}{2} s^2 b' + o(s^2) \qquad b' = \left.\frac{d^2 B(t)}{dt^2}\right|_{t=0}$$

Up to second order we can also calculate the expansion of A^{-1} and B^{-1}

$$1 = A^{-1}(s)\, A(s) \cong \left(1 + a''s + \frac{1}{2} a''' s^2\right)\left(1 + as + \frac{1}{2} a' s^2\right)$$

$$= 1 +$$

$$(a'' + a)\, s + \qquad \Rightarrow a'' = -a$$

$$\left(a''a + \frac{1}{2} a''' + \frac{1}{2} a'\right) s^2 + \qquad \Rightarrow a''' = 2a^2 - a'$$

$$o(s^2)$$

$$\Rightarrow A^{-1}(s) = 1 - as + \left(a^2 - \frac{1}{2} a'\right) s^2$$

Analogously $B(s) = 1 + bs + \frac{1}{2} b' s^2 + o(s^2)$ $B^{-1}(s) = 1 - bs + \left(b^2 - \frac{1}{2} b'\right) s^2$

We can now evaluate $C(t)$ to second order in s

$$C(t) \cong \left(\mathbb{1} + sa + \frac{1}{2}s^2 a'\right)\left(\mathbb{1} + sb + \frac{1}{2}s^2 b'\right)\left(\mathbb{1} - sa + s^2\left(a^2 - \frac{1}{2}a'\right)\right)\left(\mathbb{1} - sb + s^2\left(b^2 - \frac{1}{2}b'\right)\right)$$

$$= \mathbb{1} + s\left(\cancel{a} - \cancel{a} + \cancel{b} - \cancel{b}\right) + s^2\left(\cancel{\tfrac{1}{2}a'} + \cancel{\tfrac{1}{2}b'} + \cancel{a^2} - \cancel{\tfrac{1}{2}a'} + \cancel{b^2} - \cancel{\tfrac{1}{2}b'} + ab - \cancel{a^2} + \cancel{ab} - \cancel{ab} - ba - \cancel{\ldots}\right)$$

$$= \mathbb{1} + \underset{t}{\underset{\shortparallel}{s^2}}(ab - ba) \Rightarrow C = [a, b].$$

■

This leads immediately to the fundamental theorem

Theorem III For every linear Lie group $\mathcal{G}$ there exists a corresponding real Lie algebra $\mathcal{L}$ of the same dimension. More precisely, if $\mathcal{G}$ has dimension n then the $m \times m$ matrices $a_p = \left.\frac{\partial A}{\partial x_p}\right|_{\{x_j\}=0}$ $p = 1 \ldots n$ form a basis for $\mathcal{L}$

proof: the only thing to be proven is that if $a, b \in V = \text{span}\{a_1, \ldots a_n\}$ $\Rightarrow [a, b] \in V$. By the theorem I if $a, b \in V \Rightarrow$ they are the tangent vectors of some analytic curve in $\mathcal{G}$. Due to theorem II one can construct an analytic curve C whose tangent vector is $[a,b]$. Being $[a,b]$ the tangent vector of an analytic curve (theorem I) $[a,b] \in V$.

$\Rightarrow V = \mathcal{L}$.

Notice: $\{a_p\}$ is a basis of $\mathcal{L}$ which depends explicitly on the parametrization. Different parametrizations generate though only isomorphic Lie algebras. By convention one denotes the Lie algebras with the same name of the Lie group but lower case.

example $SO(N) \longleftrightarrow so(N)$

Example I The real Lie algebra $\mathcal{L} = su(2)$ of the linear Lie group $\mathcal{G} = SU(2)$

From the parametrization of the element $u \in SU(2)$

$$u = \begin{vmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{vmatrix} = \begin{pmatrix} \sqrt{1 - \frac{1}{4}(x_1^2+x_2^2+x_3^2)} + \frac{i}{2}x_3 & \frac{1}{2}(x_2 + i x_1) \\ -\frac{1}{2}(x_2 - i x_1) & \sqrt{1 - \frac{1}{4}(x_1^2+x_2^2+x_3^2)} - \frac{i}{2}x_3 \end{pmatrix}$$

$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad a_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which fulfill the fundamental commutation relations

$$[a_1, a_2] = -a_3 \qquad [a_2, a_3] = -a_1 \qquad [a_3, a_1] = -a_2$$

which produces the same structure constants $c_{pq}^{r} = -\varepsilon_{pqr}$ of $so(3)$.

a_1, a_2 and a_3 are all traceless antihermitian matrices

Theorem III has the following converse

Theorem IV Every real Lie algebra is isomorphic to the real Lie algebra of some linear Lie group.

(b) Relationship of the real Lie algebra $\mathcal{L}$ to the one-parameter subgroups of $\mathcal{G}$.

Theorem V Every element a of the real Lie algebra $\mathcal{L}$ of a linear Lie group $\mathcal{G}$ is associated with a one-parameter subgroup of $\mathcal{G}$ defined by

$$A(t) = \exp(ta)$$

for $-\infty < t < +\infty$.

Theorem VI Every element of a linear Lie group $\mathcal{G}$ in some small neighbourhood of its identity belongs to some one-parameter subgroup of $\mathcal{G}$. That is, every such element of $\mathcal{G}$ can be obtained by exponentiating some element of the corresponding Lie algebra.

proof: For any set of n real numbers $(x_1', x_2' \dots x_n')$ define the $m \times m$ matrix $A(x_1' \dots x_n')$ by:

$$A(x_1' \dots x_n') = \exp(x_1' a_1 + x_2' a_2 + \dots + x_n' a_n)$$

where $\{a_p\}$ is a basis for $\mathcal{L}$. According to Theorem V $A(\{x_k'\}) \in \mathcal{G}$.
$\Rightarrow$ the original parameters $x_1, \dots x_n$ can be written in terms of analytical functions $x_j = f_j(\{x_k'\})$ with the condition $f_j(\{x_k'\} = 0) = 0$.
As the Jacobian $\partial x_j / \partial x_k' \neq 0$ $\Rightarrow$ the f_j are a one-to-one mapping between small neighbourhoods of the two origins. $\Rightarrow$ Every elements of $\mathcal{G}$ close to 1_m can be expressed as the exp of an element of $\mathcal{L}$. ∎

Theorem VII If $\mathcal{G}$ is a compact linear Lie group, every element of the connected subgroup of $\mathcal{G}$ can be expressed in the form $\exp a$ for some element a in the corresponding Lie algebra $\mathcal{L}$.
Finally if $\mathcal{G}$ is connected and compact (example $SU(2)$ or $SO(3)$) every element of $\mathcal{G}$ has the form $\exp a$ for some $a \in \mathcal{L}$.

Notice

- the exponential mapping needs not to be unique even for compact connected Lie groups

- The exponential mapping provides a direct way of determining the real Lie algebras associated to an important class of linear Lie groups.

<u>Example</u> The real Lie algebra $\mathcal{L} = su(N)$ for $\mathcal{G} = SU(N)$ for $N \geq 2$.

$\exp(ta)$ is a one parameter subgroup of $\mathcal{G} = SU(N)$ $\Rightarrow$ a is a $N \times N$ matrix. $\exp(ta)$ is unitary $\Rightarrow$ $\exp(-ta) = \exp(ta^+)$

$\Rightarrow$ it follows that $a^+ = -a$. $\Rightarrow$ $a = ih$ where $h^+ = h$. $h = S\Delta S^{-1}$

$\det \exp(ith) = 1 = \det[S \exp(it\Delta) S^{-1}] = \det \exp(it\Delta) = \exp(it \operatorname{Tr}\Delta)$

$= \exp(t \operatorname{Tr} a)$ $\Rightarrow$ $\operatorname{Tr} a = 0$.

The Lie algebra of $SU(N)$ is composed of traceless anti-Hermitian $N \times N$ matrices.

10.4 <u>Representation of Lie algebras</u>

<u>Def</u> : Representation of a Lie algebra $\mathcal{L}$. Suppose that to every $a \in \mathcal{L}$ there exists a $d \times d$ matrix $\Gamma(a)$ such that

(i) $\Gamma(\alpha a + \beta b) = \alpha \Gamma(a) + \beta \Gamma(b)$

$\forall\ \alpha, \beta \in$ field of $\mathcal{L}$ (i.e. $\mathbb{C}$ or $\mathbb{R}$)

(ii) $\Gamma([a, b]) = [\Gamma(a), \Gamma(b)]$ $\forall\ a, b \in \mathcal{L}$

$\Rightarrow$ the set of matrices $\Gamma(a)$ form a d-dimensional representation of $\mathcal{L}$.

Notice:

- The set of matrices $\Gamma(a)$ form themselves a Lie algebra
- The mapping $a \to \Gamma(a)$ is an homomorphism between Lie algebras which is specified once it is given for a basis a_k.
- Many of the ideas on representations presented for (finite) group also apply to Lie algebras.

In particular one can state Schur's Lemma 1 + 2 for Lie algebras

Theorem I Let Γ and Γ' be two irreducible representations of a Lie algebra $\mathcal{L}$ of dimensions d and d' respectively and suppose that there exists a $d \times d'$ matrix M such that $\Gamma(a) M = M \Gamma'(a)$ $\forall a \in \mathcal{L}$ $\Rightarrow$ either $A = 0$ or $d = d'$ and $\det M \neq 0$ $\Rightarrow$ Γ and Γ' are similar. (i.e. $\Gamma(a) = M\Gamma'(a) M^{-1}$).

proof: Contrary to the corresponding proof for groups, Γ are not unitary

Let us assume $d \geq d'$ and $\psi_1 \dots \psi_d$ a basis in the vector space V in which operate $\Phi(a)$ the linear operator associated to a by the homomorphism Φ, such that

$$\Phi(a) \psi_j = \sum_{i=1}^{d} \Gamma(a)_{ij} \psi_i$$

Moreover let us introduce the functions $\phi_k = \sum_{j=1}^{d} M_{jk} \psi_j$ $k = 1 \dots d'$.
$S = \text{span}(\phi_k)$ is a subspace of V. $\Rightarrow$ we can construct

$$\Phi(a) \phi_k = \Phi(a) \sum_{j=1}^{d} M_{jk} \psi_j = \sum_{ij=1}^{d} M_{jk} \Gamma_{ij}(a) \psi_i$$

$$= \sum_{i=1}^{d} (\Gamma M)_{ik} \psi_i = \sum_{i=1}^{d} (M \Gamma')_{ik} \psi_i$$

$$= \sum_{i=1}^{d} \sum_{l=1}^{d'} M_{il} \Gamma'_{lk} \psi_i = \sum_{l=1}^{d'} \Gamma'_{lk} \phi_l \quad \in S$$

$\Rightarrow$ S is an invariant subspace of V $\forall\, a \in \mathcal{L}$. But Γ is irreducible $\Rightarrow$ V has no invariant subspaces smaller than itself.

$\Rightarrow$ either $\phi_k = 0$ $\forall\, k = 1, \dots, d'$ $\Rightarrow$ $M = 0$ or $V = S$. In the latter case either ϕ_k are linearly independent $\Rightarrow$ $d' = d$ and $\det M \neq 0$ or ϕ_k are not linearly independent $\Rightarrow$ $d' > d$ ↯. Clearly the only compatible possibilities are $M = 0$ or $d = d'$ $\det M \neq 0$.

<u>Theorem II</u> if Γ is a d-dimensional irrep of a Lie algebra $\mathcal{L}$ and M is a $d \times d$ matrix such that $\Gamma(a) M = M \Gamma(a)$ $\forall\, a \in \mathcal{L}$

$\Rightarrow$ $M = \lambda \mathbb{1}$.

<u>proof</u>: Let $M' = M - \mu \mathbb{1}$ where the complex number μ is chosen such that $\det(M')$ {i.e. polinomial of order d in μ} vanishes.

$\Rightarrow$ $\Gamma M' = M' \Gamma = 0$ and the only possibility from theorem I is $M' = 0$ $\Rightarrow$ $M = \mu \mathbb{1}$.

Of particular importance is the connection between the representation of a linear Lie group and those of its corresponding real Lie algebra.

<u>Theorem III</u> Let $\Gamma_{\mathcal{G}}$ be a d-dimensional analytic representation of a linear Lie group $\mathcal{G}$, whose corresponding real Lie algebra is $\mathcal{L}$.

a) There exists a d-dimensional representation $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$ defined $\forall\, a \in \mathcal{L}$ as

$$\Gamma_{\mathcal{L}}(a) = \left[\frac{d}{dt} \Gamma_{\mathcal{G}}(\exp(ta))\right]_{t=0}$$

b) For all $a \in \mathcal{L}$ and all real t

$$\exp\{t \Gamma_{\mathcal{L}}(a)\} = \Gamma_{\mathcal{G}}(\exp(ta))$$

c) If $\Gamma_{\mathcal{G}}$ and $\Gamma'_{\mathcal{G}}$ are two d-dimensional analytic representations of $\mathcal{G}$ and $\Gamma_{\mathcal{L}}$ and $\Gamma'_{\mathcal{L}}$ are the associated representations of $\mathcal{L}$ defined as in a) $\Rightarrow$ $\Gamma_{\mathcal{L}}$ is equivalent to $\Gamma'_{\mathcal{L}}$ if $\Gamma_{\mathcal{G}}$ is equivalent to $\Gamma'_{\mathcal{G}}$.

d) $\Gamma_{\mathcal{G}}$ reducible $\Rightarrow$ $\Gamma_{\mathcal{L}}$ reducible. If $\mathcal{G}$ connected also $\Gamma_{\mathcal{L}}$ red $\Rightarrow$ $\Gamma_{\mathcal{G}}$ red.

e) $\Gamma_{\mathcal{L}}$ irreducible $\Rightarrow$ $\Gamma_{\mathcal{G}}$ irreducible.

f) If $\Gamma_{\mathcal{G}}$ is a unitary representation $\Rightarrow$ $\Gamma_{\mathcal{L}}(a)$ is anti-Hermitian $\forall\, a \in \mathcal{L}$

The converse statements of c) - f) are true if $\mathcal{G}$ is connected.

proof:

a) The analyticity of $\Gamma_{\mathcal{G}}$ ensures that the derivative can be taken. We have to prove that $\Gamma_{\mathcal{L}}(a)$ is homomorphic i.e.

i) $\Gamma_{\mathcal{L}}(\alpha a) = \alpha \Gamma_{\mathcal{L}}(a)$ $\quad \forall\, \alpha$ real

ii) $\Gamma_{\mathcal{L}}(a+b) = \Gamma_{\mathcal{L}}(a) + \Gamma_{\mathcal{L}}(b)$ $\quad \forall\, a, b \in \mathcal{L}$

iii) $\Gamma_{\mathcal{L}}([a,b]) = [\Gamma_{\mathcal{L}}(a), \Gamma_{\mathcal{L}}(b)]$ $\quad \forall\, a, b \in \mathcal{L}$.

i) $$\Gamma_{\mathcal{L}}(\alpha a) = \frac{d}{dt} \Gamma_{\mathcal{G}} \exp(t\alpha a) \overset{s=\alpha t}{=} \alpha \frac{d}{ds}\left[\Gamma_{\mathcal{G}} \exp(sa)\right]_{s=0} = \alpha \Gamma_{\mathcal{L}}(a)$$

ii)

$$\Gamma_{\mathcal{L}}(a+b) = \frac{d}{dt}\left[\Gamma_{\mathcal{G}}\left(\exp\left(t(a+b)\right)\right)\right]_{t=0} =$$

$$= \lim_{t\to 0}\frac{1}{t}\left[\Gamma_{\mathcal{G}}\left(\exp(t(a+b))\right) - \overbrace{\Gamma_{\mathcal{G}}(\mathbb{1})}^{\mathbb{1}_d}\right]$$

But, to first order in t $\exp(t(a+b)) = \exp(ta)\exp(tb) + o(t)$

$$\Gamma_{\mathcal{G}}\left[\exp(ta)\exp(tb)\right] = \Gamma_{\mathcal{G}}\left[\exp(ta)\right]\Gamma_{\mathcal{G}}\left[\exp(tb)\right] =$$

$$= \left(\mathbb{1}_d + \Gamma_{\mathcal{L}}(a)\,t\right)\left(\mathbb{1}_d + \Gamma_{\mathcal{L}}(b)\,t\right) + o(t)$$

Putting all together:

$$\Gamma_{\mathcal{L}}(a+b) = \lim_{t\to 0}\frac{1}{t}\left[\left(\mathbb{1}_d + \Gamma_{\mathcal{L}}(a)\,t\right)\left(\mathbb{1}_d + \Gamma_{\mathcal{L}}(a)\,t\right) - \mathbb{1}_d + o(t)\right] =$$

$$= \Gamma_{\mathcal{L}}(a) + \Gamma_{\mathcal{L}}(b).$$

iii) $$\left[\Gamma_{\mathcal{L}}(a), \Gamma_{\mathcal{L}}(b)\right] = \lim_{s\to 0}\lim_{t\to 0}\left\{\left[\Gamma_{\mathcal{G}}(\exp(ta)) - \mathbb{1}_d, \Gamma_{\mathcal{G}}(\exp(tb)) - \mathbb{1}_d\right]\right\}\frac{1}{st}$$

$$= \lim_{s\to 0}\lim_{t\to 0}\left\{\left[\Gamma_{\mathcal{G}}(\exp(ta)), \Gamma_{\mathcal{G}}(\exp(sb))\right]\right\}\frac{1}{st}$$

$$\overset{\substack{\Gamma_{\mathcal{G}}\text{ is a}\\ \text{repr.}}}{=} \lim_{s\to 0}\lim_{t\to 0}\left[\Gamma_{\mathcal{G}}\left(\exp(ta)\exp(sb)\right) - \Gamma_{\mathcal{G}}\left(\exp(sb)\exp(ta)\right)\right]\frac{1}{st}$$

$$= \lim_{s\to 0}\lim_{t\to 0}\left[\Gamma_{\mathcal{G}}\left(\exp\left(ta+sb+\tfrac{1}{2}[a,b]st+\dots\right)\right) - \Gamma_{\mathcal{G}}\left(\exp\left(sb+ta+\tfrac{1}{2}[b,a]+\dots\right)\right)\right]\frac{1}{st}$$

$$= \lim_{s\to 0}\lim_{t\to 0}\left[\cancel{\mathbb{1}_d} + \Gamma_{\mathcal{L}}\left(ta+sb+\tfrac{1}{2}[a,b]st\right) - \cancel{\mathbb{1}_d} - \Gamma_{\mathcal{L}}\left(sb+ta+\tfrac{1}{2}\underbrace{[b,a]}_{=-[a,b]}\right)\right]\frac{1}{st}$$

$$\overset{i)-ii)}{=} \lim_{s\to 0}\lim_{t\to 0}\left[\cancel{t\Gamma_{\mathcal{L}}(a)} + \cancel{s\Gamma_{\mathcal{L}}(b)} + \tfrac{1}{2}st\,\Gamma_{\mathcal{L}}([a,b]) - \cancel{s\Gamma_{\mathcal{L}}(b)} - \cancel{t\Gamma_{\mathcal{L}}(a)} + \tfrac{1}{2}st\,\Gamma_{\mathcal{L}}([a,b])\right]\frac{1}{st}$$

$$= \Gamma_{\mathcal{L}}([a,b]).$$

b) Since $\Gamma_{\mathcal{G}}$ is an homomorphism

$$\Gamma_{\mathcal{G}}\left[\exp((s+t)a)\right] = \Gamma_{\mathcal{G}}\left[\exp(sa)\right]\Gamma_{\mathcal{G}}\left[\exp(ta)\right]$$

$\forall\, s,t \in \mathbb{R}$ $\forall\, a \in \mathcal{L}$. If $\Gamma_{\mathcal{G}}\left[\exp(ta)\right] = \mathbb{1}_d$ $\forall t \Rightarrow \Gamma_{\mathcal{L}}(a) = 0$ and the equation that we want to prove holds. On the other hand, if $\Gamma_{\mathcal{G}}(\exp(ta)) \neq \mathbb{1}$ for some interval $t \in [0, t_0) \Rightarrow \Gamma_{\mathcal{G}}(\exp(ta))$ form a one-parameter subgroup of the representation $\Gamma_{\mathcal{G}}$. $\Rightarrow \Gamma_{\mathcal{L}}(a)$ is the tangent vector of an analytic curve and $\Rightarrow$ element of the Lie algebra of $\Gamma_{\mathcal{G}}$. Since $\Gamma_{\mathcal{L}}(a)$ is also the tangent vector of the analytic curve/ one parameter subgroup of $\Gamma_{\mathcal{G}}$ $\exp[t\,\Gamma_{\mathcal{L}}(a)]$ it follows that

$$\Gamma_{\mathcal{G}}\left[\exp(ta)\right] = \exp(t\,\Gamma_{\mathcal{L}}(a))$$

i.e the two one parameter subgroups coincide.

c) Suppose that $\Gamma'_{\mathcal{G}}(A) = S^{-1}\Gamma_{\mathcal{G}}(A)S$ $\forall A \in \mathcal{G}$

$$\Gamma'_{\mathcal{L}}(a) = \left[\frac{d}{dt}\left\{S^{-1}\Gamma_{\mathcal{G}}(\exp(ta))S\right\}\right]_{t=0} = \left[\frac{d}{dt}\left\{S^{-1}\exp(t\,\Gamma_{\mathcal{L}}(a))S\right\}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\exp(t\,S^{-1}\Gamma_{\mathcal{L}}(a)S)\right\}\right]_{t=0} = S^{-1}\Gamma_{\mathcal{L}}(a)S.$$

Conversely if $\Gamma'_{\mathcal{L}}(a) = S^{-1}\Gamma_{\mathcal{L}}(a)S$ $\Rightarrow$ from b) $\Gamma'_{\mathcal{G}}(\exp(ta)) = S^{-1}\Gamma_{\mathcal{G}}(\exp(ta))S$ which extends to the whole $\mathcal{G}$ if this is connected.

d) The equation in b) implies that $\Gamma_{\mathcal{G}}$ red $\Rightarrow \Gamma_{\mathcal{L}}$ reducible. The converse is true thanks to b) if $\mathcal{G}$ is connected since every element of $\mathcal{G}$ can be written in exponential form.

e) It's a direct consequence of d) since $A \Rightarrow B \Rightarrow \neg B \Rightarrow \neg A$

f) It follows from eq. in b) since $\Gamma_{\mathcal{G}}^{-1}(A) = \Gamma_{\mathcal{G}}^{+}(A)$ if $\Gamma_{\mathcal{G}}$ is unitary but from eq. in b)

$$\Gamma_{\mathcal{G}}^{-1}(A) = \exp(-t\,\Gamma_{\mathcal{L}}(a)) = \exp(t\,\Gamma_{\mathcal{L}}^{+}(a)) =$$

$$= \exp(t\,\Gamma_{\mathcal{L}}(a))^{+} = \Gamma_{\mathcal{G}}(A)^{+}.$$

■

<u>Notice</u> This theorem does <u>not</u> implie that every representation $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$ gives a representation of $\mathcal{G}$ by exponentiation. Rather: if $\Gamma_{\mathcal{G}}$ is analytic $\Rightarrow$ $\Gamma_{\mathcal{L}}$ is defined and $\Gamma_{\mathcal{G}}$ can be obtained by exponentiation of $\Gamma_{\mathcal{L}}$.

Example: Connection between the representations of $\mathcal{L} = so(2)$ and $\mathcal{G} = SO(2)$

$\mathcal{L} = so(2)$ is one-dimensional

$$a_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The only relevant commutation relation is the trivial one $[a_1, a_1] = 0$

$\Rightarrow$ $\Gamma_{\mathcal{L}}(a_1) = p$ is a one-dimensional representation of $\mathcal{L}$ $\forall p \in \mathbb{C}$.

$\exp\{t\,\Gamma_{\mathcal{L}}(a_1)\} = \exp(tp)$, While the elements of $\mathcal{G}$ have the form

$$\exp(ta_1) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

However $\exp((t+2\pi)a_1) = \exp(ta_1)$ but $\exp\{(t+2\pi)\,\Gamma_{\mathcal{L}}(a_1)\} = \exp(2\pi p)\exp(t\,\Gamma_{\mathcal{L}}(a_1))$

$\Rightarrow$ this representation $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$ gives a repr. of $SO(2)$ only if $p = iq$ where q is some integer.

Theorem IV If G is a compact Lie group, then every continuous representation is analytic and viceversa.

10.5 Irreducible representations of the Lie algebras su(2) and so(3)

We have already illustrated the intimate connection between the Lie algebra so(3) and the angular momentum operators. Moreover su(2) and so(3) are isomorphic => we expect them to share a set of irreducible representations.
The basis elements:

$$a_1 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \qquad a_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad a_3 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

are linearly independent over the field of complex numbers (verify), thus the complexification $\tilde{\mathcal{L}}$ of su(2) can be taken simply as the set of complex linear combinations of a_i $i=1,2,3$.
As the Lie group SU(2) is compact, each irreducible representation of SU(2) can be taken to be unitary. => the corresp. irreducible representation of $\tilde{\mathcal{L}}$ consists of anti-Hermitian matrices.

$\psi_1 \dots \psi_d$ is an orthonormal basis of a vector space V with scalar product.

$$\forall a \in su(2) \qquad \Phi(a)\psi_p = \sum_{q=1}^{d} \Gamma(a)_{qp}\psi_q \qquad p = 1 \dots d$$

$$(\psi_q, \Phi(a)\psi_p) = \Gamma(a)_{qp} = -\Gamma(a)^*_{pq} = -(\Phi(a)\psi_q, \psi_p)$$ it follows that

$$(\phi, \Phi\psi) = -(\Phi(a)\phi, \psi)$$

$\forall\ \phi, \psi \in V$ and $\forall\ a \in \tilde{su}(2)$. It is convenient to define 3 linear operators

$$A_1, A_2, A_3 \quad \text{by} \quad A_p = -i\,\Phi(a_p)$$

such that

$$(\phi, A_p \psi) = (A_p \phi, \psi)$$

i.e. A_p are self-adjoint operators. Since Φ is an homomorphism, it conserves the Lie product $[\Phi(a_p), \Phi(a_q)] = \Phi([a_p, a_q])$ and we obtain

$$[A_p, A_q] = i \sum_{r=1}^{3} \varepsilon_{pqr} A_r \qquad p, q = 1,2,3$$

where ε_{pqr} is the Levi-Civita symbol. Within the complex algebra $\hat{\mathcal{L}}$ we define the "ladder operators"

$$A_\pm = A_1 \pm i A_2$$

so that $\quad A_1 = \frac{1}{2}(A_+ + A_-) \quad$ and $\quad A_2 = -\frac{1}{2} i (A_+ - A_-)$

We obtain thus the following commutation relations:

$$[A_3, A_\pm] = [A_3, A_1 \pm i A_2] = i A_2 \pm i(-i) A_1 = \pm A_\pm$$

$$[A_+, A_-] = [A_1 + iA_2, A_1 - iA_2] = 2 A_3$$

Moreover A_+ and A_- are one the adjoint of the other. Finally, we can define the operator $A^2 = A_1^2 + A_2^2 + A_3^2$ for which we obtain

$$[A^2, A_p] = 0 \qquad p = 1,2,3$$

which in turns implies

$$[A^2, A_\pm] = 0$$

Finally one can consider the product $A_\mp A_\pm$ and obtain

$$A_\mp A_\pm = (A_1 \mp iA_2)(A_1 \pm iA_2) = A_1^2 + A_2^2 \pm i [A_1, A_2] = A^2 - A_3^2 \mp A_3$$

$$[A^2, A_p] = \sum_q [A_q^2, A_p] = \sum_q \left(A_q [A_q, A_p] + [A_q, A_p] A_q \right)$$

$$= i \sum_{qr} \left(\varepsilon_{qpr} A_q A_r + \varepsilon_{qpr} A_r A_q \right) = i \sum_{qr} A_q A_r \left(\varepsilon_{qpr} + \varepsilon_{rpq} \right) = 0$$

where the last result is obtained since $\varepsilon_{qpr} = - \varepsilon_{rpq}$.

The operators of the angular momentum in quantum mechanics fulfill the relations

$$[J_x, J_y] = i\hbar J_z \qquad [J_y, J_z] = i\hbar J_x \qquad [J_z, J_x] = i\hbar J_y$$

$\Rightarrow$ the identification $\quad J_x = \hbar A_1 \quad J_y = \hbar A_2 \quad J_z = \hbar A_3$

$$J_+ = \hbar A_+ \qquad J_- = \hbar A_-$$

$$J^2 = \hbar^2 A^2$$

Since $[A^2, A_p] = 0$ $\quad [\Gamma(A^2), \Gamma(a)] = 0 \quad \forall\, a \in su(2)$ $\Rightarrow$ if Γ is irreducible $\Gamma(A^2) = \lambda \mathbb{1}_d$ where d is the dimension of the irreducible representation.
$\Rightarrow$ all the basis vectors $\psi_1 \dots \psi_d$ of V are eigenvectors of A^2 with the same eigenvalue.

Theorem I   To every non-negative integer or half-integer j (i.e. $j = 0, \frac{1}{2}, 1, \frac{3}{2} \dots$) there exists an irreducible representation of $su(2)$ (and its complexification $\tilde{\mathcal{L}}$) of dimension $d = 2j+1$. The orthonormal basis vectors of the irrep specified by j may be denoted by ψ^j_m where $m = -j, -j+1 \dots j$. Each basis vector may be chosen to be a simultaneous eigenvector of A^2 and A_3 with eigenvalues $j(j+1)$ and m respectively

$$A^2 \psi^j_m = j(j+1)\psi^j_m$$

$$A_3 \psi^j_m = m\,\psi^j_m$$

Moreover, the relative phases of the basis vectors may be chosen so that

$$A_+ \psi^j_m = \{(j-m)(j+m+1)\}^{1/2} \psi^j_{m+1}$$

$$A_- \psi^j_m = \{(j+1)(j-m+1)\}^{1/2} \psi^j_{m-1}$$

Up to equivalence these are the only irrep of $su(2)$. The matrices of these irrep may be denoted by $D^j(a)$ [D from "Darstellung", represen tation in German] with elements $D^j(a)_{mm'}$ $a \in su(2)$. In particular

$$D^j(a_1)_{mm'} = \tfrac{1}{2} i \left[\delta_{m'm+1}\{(j-m)(j+m+1)\}^{1/2} + \delta_{m',m-1}\{(j+m)(j-m+1)\}^{1/2} \right]$$

$$D^j(a_2)_{mm'} = \tfrac{1}{2} \left[\delta_{m'm+1}\{(j-m)(j+m+1)\}^{1/2} - \delta_{m'm-1}\{(j+m)(j-m+1)\}^{1/2} \right]$$

$$D^j(a_3)_{mm'} = i m \, \delta_{m'm}$$

proof

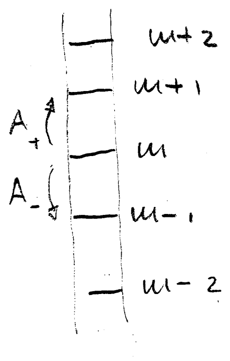
Since $[A^2, A_3] = 0 \Rightarrow$ the basis of an irrep of su(2) can be chosen to be an eigenbasis for A_3. Let ψ be an element of the basis

$$A_3 \psi = m \psi$$

As A_3 is self-adjoint $\Rightarrow$ $m \in \mathbb{R}$.

$$A_3 A_+ \psi = A_+ A_3 \psi + A_+ \psi = A_+ m\psi + A_+\psi = (m+1) A_+ \psi$$
$$A_3 A_- \psi = A_- A_3 \psi - A_- \psi = A_- m\psi - A_-\psi = (m-1) A_- \psi$$

$\Rightarrow$ $A_\pm \psi$ are eigenstates of A_3 with eigenvalue $m \pm 1$ unless $A_\pm \psi = 0$.



The operators A_+ and A_- are called ladder operators since they connect neighbouring eigenstates with equidistant A_3 eigenvalues...

Let us assume a finite dimensional irrep of su(2) $\Rightarrow$ $\exists$ an eigenstate of A_3 with maximum eigenvalue j.

It follows immediately that $A_+ \varphi_j = 0$ otherwise $j+1$ would be the eigenvalue of φ_{j+1} in the SAME irrep.

$$A_+ \varphi_j = 0 \quad \Rightarrow 0 = A_- A_+ \varphi_j = \left(A^2 - A_3^2 - A_3\right) \varphi_j = \left(A^2 - j^2 - j\right) \varphi_j$$

$$\Rightarrow A^2 \varphi_j = j(j+1)\, \varphi_j$$

All elements of the irrep <u>must</u> have $j(j+1)$ as their eigenvalue for A^2.
Now consider the set of vectors $\{A_- \varphi_j,\ A_-^2 \varphi_j, \dots\}$ which correspond to the set of eigenvalues $j-1, j-2, \dots$ for A_3. As we are looking for a finite dimensional irrep $\Rightarrow$ there exists an eigenvector associated to the minimum eigenvalue: $j-k \leftrightarrow A_-^k \varphi_j$.

$$A_-^{k+1} \varphi_j = 0 \quad \Rightarrow 0 = A_+ A_- A_-^k \varphi_j = \left(A^2 - A_3^2 + A_3\right) A_-^k \varphi_j =$$

$$= \left[j(j+1) - (j-k)^2 + j - k\right] \varphi_{j-k}$$

Since $\varphi_{j-k} \neq 0 \Rightarrow$
$$\cancel{j^2} + j - \cancel{j^2} + 2kj - k^2 + j - k = 2j - k + 2kj - k^2 =$$

$$= (k+1)(2j-k) \quad \Rightarrow \quad \boxed{j = \frac{k}{2}}$$

Thus the only possible values of j are $j = 0, \frac{1}{2}, 1, \frac{3}{2}$. Moreover the minimum eigenvalue of A_3 is $j-k = j - 2j = -j \Rightarrow$ the dimensionality of the irrep is $d = 2j+1$.
Let φ_m^j denote the simultaneous eigenvector of A_3 and A^2 with eigenvalue m and $j(j+1)$ respectively. $\Rightarrow$ for $m = j, j-1, \dots, -j+1$ as $A_- \varphi_m^j$ is an eigenstate of A_3 with eigenvalue $m-1$

$$A_- \varphi_m^j = \mu_m^j \varphi_{m-1}^j$$

As φ_m^j and φ_{m-1}^j are assumed to be normalized

$$|\mu_m^j|^2 = (\mu_m^j \psi_{m-1}^j, \mu_m^j \psi_{m-1}^j) = (A_- \psi_m^j, A_- \psi_m^j) = (\psi_m^j, A_+ A_- \psi_m^j)$$

But $A_+ A_- \psi_m^j = (A^2 - A_3^2 + A_3)\psi_m^j = [j(j+1) - m(m-1)]\psi_m^j$. It follows immediately that

$$|\mu_m^j|^2 = j(j+1) - m(m-1) = (j+m)(j-m+1)$$

The phase can be chosen such that μ_m^j is real. For what concern the operator A_+

$$A_+ \psi_{m-1}^j = 1/\mu_m^j\, A_+ A_- \psi_m^j = \frac{\mu_m^{j\,2}}{\mu_m^j} A_m^j = \mu_m^j A_m^j .$$

Thus, summarizing

$$A_- \psi_m^j = \mu_m^j \psi_{m-1}^j = \{(j+m)(j-m+1)\}^{1/2} \psi_{m-1}^j$$

$$A_+ \psi_m^j = \mu_{m+1}^j \psi_{m+1}^j = \{(j-m)(j+m+1)\}^{1/2} \psi_{m+1}^j$$

For $m = -j \cdots j$, As $\mu_{j+1}^j = 0$. As $a_1 = \frac{1}{2} i (A_+ + A_-)$ and $a_2 = \frac{1}{2}(A_+ - A_-)$ the equations for the representation are obtained immediately. ∎

Notice that this proof made use of the properties of A^2 which is not a member of $\tilde{\mathcal{L}}$ since it contains the <u>product</u> of members of $\tilde{\mathcal{L}}$.

Let us consider the special cases $j = 0, 1/2, 1$. For $j=0$ the irreducible representation is 1 dimensional and $D^0(a) = 0 \ \forall\, a \in su(2)$.

For $j = 1/2$ $D^{1/2}(a_j) = a_j$ i.e. the 2 dimensional irrep. here identified is identical to the defining two-dimensional representation of $su(2)$.

For $j = 1$ we obtain the 3 dimensional representation

$$\Delta^1(a_1) = \frac{1}{2} i \begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}, \quad \Delta^1(e_2) = \frac{1}{2} \begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \quad \Delta^1(e_3) = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}$$

This rep. is equivalent (but not identical) to that given by the antisymmetric matrices.

10.6 Universal covering groups

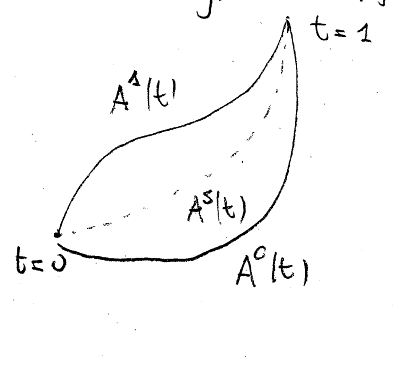
In order to understand the intimate connection between representation of Lie algebras and Lie groups we have to introduce the concept of universal covering. This is obtained in steps.

The most important property of a covering group is that it is SIMPLY CONNECTED.

Let's consider $\mathcal{G}$ a connected linear Lie group of dimension n whose elements are $m \times m$ matrices A parametrized by n real parameter $y_1 \dots y_n$. $f_1^0(t), \dots, f_n^0(t)$ are n continuous functions of $t \in [0,1]$.

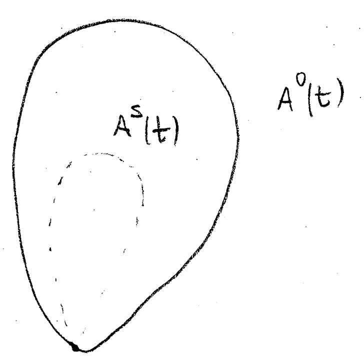
$A^0(t) = A(f_1^0(t), \dots, f_n^0(t))$ is a path in $\mathcal{G}$ with initial point $A^0(0)$ and final point $A^0(1)$. Analogously $A^1(t)$ is a second path associated with $\{f_n^1(t)\}$ and same initial and final points $A^0(0) = A^1(0)$, $A^0(1) = A^1(1)$

The two paths are homotopic if one can transform continuously one into another. i.e $\exists$ $h_j(t,s)$ continuous in s, and t such that $h_j(t,0) = f_j^0(t)$ and $h_j(t,1) = f_j^1(t)$



In particular, if $A^0(0) = A^0(1)$ => the path is said to be a loop with base point in $A^0(0)$.

If $A^0(t)$ is a loop with base point $A^0(0)$ and $A^0(t)$ is homotopic to the constant loop $A^1(t) = A^0(0)$ $\forall\ t \in [0,1]$ => $A^0(t)$ is contractible to the point $A^0(0)$.



$t=0$ and $t=1$ $\quad A^1(t) = A^0(0) \quad t \in [0,1]$

Definition Simply connected linear Lie group $\mathcal{G}$.

A connected linear Lie group $\mathcal{G}$ is said to be "simply connected" if every loop in $\mathcal{G}$ is contractible to a point.

Example I $SU(2)$ as a simply connected group

A typical element of $SU(2)$ is given by $u = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$

which, with $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ implies $\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$.

Thus we have the equation of a sphere in 4D. Every point on this sphere correspond to a different element of $SU(2)$. Loops with base on $\mathbb{1}_2$ in $SU(2)$ correspond to paths on the "parameter sphere" starting and ending in $(1,0,0,0)$. => they can be contracted to $(1,0,0,0)$.

Example II $SO(3)$ as a non simply connected group: Recall the discussion with the parameter ball and path with and without jumps. (pages 80-83 of these notes).

One more concept is required before we can continue with the theory.

Definition: The centre $\mathcal{Z}$ of a group $\mathcal{G}$

The "centre" of a group $\mathcal{G}$ is the subgroup $\mathcal{Z}$ consisting of all elements of $\mathcal{G}$ with each commute with every element of $\mathcal{G}$.

- $\mathcal{Z}$ is not empty since it contains at least the identity
- Every subgroup of $\mathcal{Z}$ is Abelian and is an invariant subgroup of $\mathcal{G}$.

$\mathcal{Z}$ is the "central invariant subgroup" of $\mathcal{G}$.

Theorem I (only stated) If $\mathcal{G}$ is a connected Lie group $\Rightarrow$ there exists a simply connected Lie group $\tilde{\mathcal{G}}$ (unique up to isomorphism) such that:

a) $\mathcal{G}$ is analytically isomorphic to a factor group $\tilde{\mathcal{G}}/\mathcal{K}$ where $\mathcal{K}$ is a discrete central invariant subgroup of $\tilde{\mathcal{G}}$

b) If $\mathcal{G}$ is simply connected $\Rightarrow$ $\mathcal{G}$ is isomorphic to $\tilde{\mathcal{G}}$

c) The real Lie algebras of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are isomorphic

d) every representation of the real Lie algebra of $\tilde{\mathcal{G}}$ is associated with a representation of $\tilde{\mathcal{G}}$ by:

$$\Gamma_{\tilde{\mathcal{L}}}(a) = \left[\frac{d}{dt}\, \Gamma_{\tilde{\mathcal{G}}}(\exp(ta))\right]_{t=0}$$

Comment: The factor group $\mathcal{G}/\mathcal{S}$ is the set of the right cosets of an invariant subgroup $\mathcal{S}$, with the multiplication defined as:

$$(\mathcal{S}T_1)\cdot(\mathcal{S}T_2) := \mathcal{S}T_1T_2$$

$\tilde{\mathcal{G}}$ is unique up to isomorphism and is called the universal covering of $\mathcal{G}$.

Each representation $\Gamma_{\tilde{\mathcal{G}}}$ of $\tilde{\mathcal{G}}$ is also a representation $\Gamma_{\mathcal{G}}$ of $\mathcal{G}$ with the prescription

$$\Gamma_{\mathcal{G}}(T) = \Gamma_{\tilde{\mathcal{G}}}(KT)$$

$\forall\, T \in \mathcal{G}$ if and only if $\Gamma_{\tilde{\mathcal{G}}}(S) = 1 \quad \forall\, S \in K$.

Theorem II (only stated) If the real algebras $\mathcal{L}_1$ and $\mathcal{L}_2$ of two simply connected Lie groups $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ are isomorphic $\Rightarrow$ $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ are analytically isomorphic.

The two theorem above imply the very important result that to each real Lie algebra $\mathcal{L}$ there exists a simply connected Lie group $\tilde{\mathcal{G}}$ which is unique such that every Lie group $\mathcal{G}$ having $\mathcal{L}$ as its real Lie algebra is isomorphic to $\tilde{\mathcal{G}}/K$ K being a discrete central invariant subgroup of $\tilde{\mathcal{G}}$. $\Rightarrow$ $\tilde{\mathcal{G}}$ may be referred to as the "universal covering group of $\mathcal{L}$."

<u>Example</u> $SU(2)$, its universal covering group and the other compact Lie groups formed from it that possesses isomorphic Lie algebras.

$SU(2)$ is simply connected $\Rightarrow$ it is its own universal covering group

If $u \in \mathcal{Z}$ of $SU(2)$ and the set of 2×2 matrices with det $= 1$ is an irrep of $SU(2)$ $\Rightarrow$ (Schur's lemma) $u = \lambda \mathbb{1}_2$ but $\det u = \lambda^2$ $\Rightarrow$ $\lambda = \pm 1$, i.e. $\mathcal{Z} = \{\mathbb{1}_2, -\mathbb{1}_2\}$. $\Rightarrow$ the set of compact Lie groups with real algebras isomorphic to $su(2)$ are $SU(2)$ and $SU(2)/\mathcal{Z} = SO(3)$.

The basis functions of the representations D^j $(j=0,1,2,\dots)$ of $SO(3)$ are basis functions of the corresponding D^j representation of the Lie algebra $so(3)$

$$\hat{a}\,\psi^j_m(r) = \sum_{m'=-j}^{j} D^j(a)_{m'm}\,\psi^j_{m'}(r)$$

$\forall\, a \in so(3)$ and $m=-j,-j+1,\dots,j$. In particular one obtains:

$$-\left(\hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2\right)\psi^j_m = j(j+1)\,\psi^j_m(r)$$

$$-i\,\hat{a}_3\,\psi^j_m(r) = m\,\psi^j_m(r) \qquad (*)$$

and

$$-i\left(\hat{a}_1 - i\hat{a}_2\right)\psi^j_m(r) = \{(j+m)(j-m+1)\}^{1/2}\,\psi^j_{m-1}(r)$$

It is convenient to write the action of the operators $\hat{a}_1\ \hat{a}_2\ \hat{a}_3$ in spherical polar coordinates. (check)

$$\hat{a}_1 = -\sin\phi\,\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\,\frac{\partial}{\partial\phi}$$

$$\hat{a}_2 = \cos\phi\,\frac{\partial}{\partial\theta} - \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}$$

$$\hat{a}_3 = \frac{\partial}{\partial\phi}$$

so that the previous equations $(*)$ become:

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\psi^j_m(r) = j(j+1)$$

$$-i\,\frac{\partial}{\partial\phi}\,\psi^j_m(r) = m\,\psi^j_m(r) \qquad (**)$$

$$e^{-i\phi}\left\{-\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\phi}\right\}\psi^j_m(r) = \{(j+m)(j-m+1)\}^{1/2}\,\psi^j_{m-1}(r)$$

10.7 Representations of the Lie groups SU(2), SO(3) and O(3)

As SU(2) is the universal covering group corresponding to the Lie algebra su(2) $\Rightarrow$ every irrep of su(2) exponentiates to give an irrep of SU(2).

Let us verify it explicitly. $r^T = (z_1, z_2)$ where z_1 and $z_2 \in \mathbb{C}$. $\forall u \in$ SU(2) we define the linear operator $\hat{u}$ acting of $f(r)$:

$$\hat{u} f(r) = f(u^{-1} r)$$

Since $\widehat{(u_1 u_2)} = \hat{u}_1 \cdot \hat{u}_2 \quad \forall\ u_1, u_2 \in SU(2) \Rightarrow$ we have created an homomorphism to a space of linear operators acting on functions. Let us now take $j = 0, \frac{1}{2}, 1, \frac{3}{2} \dots$ and $\forall\ m = j, j-1 \dots, -j$ define the function

$$\psi^j_m(r) = \left[(-1)^m \{(j-m)!\,(j+m)!\}^{-\frac{1}{2}}\right] z_1^{\,j-m} z_2^{\,j+m}$$

i.e All polynomials of order $2j$ in z_1 or z_2. $\forall\ u \in SU(2)$ we let $\hat{u}$ act on ψ^j_m:

$$u = \begin{vmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{vmatrix} \quad \Rightarrow \quad u^{-1} = u^{+} = \begin{vmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{vmatrix}$$

$$\hat{u}\, \psi^j_m(r) = \left[(-1)^m \{(j-m)!\,(j+m)!\}^{-\frac{1}{2}}\right] \left(\alpha^* z_1 - \beta z_2\right)^{j-m} \left(\beta^* z_1 + \alpha z_2\right)^{j+m}$$

$$= \sum_{m'=-j}^{j} D^j_g(u)_{m'm}\, \psi^j_{m'}$$

where the last equality results from the definition of ψ^j_m, since the polynomial on the LHS is of order $2j$. In order to see that $D^j_g(u)$ is the exponential of the representation D^j of su(2), we introduce the linear operators $\hat{a}$.

$$\hat{a} f(r) = \lim_{t \to 0} f\left(\{\exp(-ta)\} r\right) - f(r)]/t = -r^T a^T \operatorname{grad} f(r)$$

where $\quad \operatorname{grad} := \begin{pmatrix} \dfrac{\partial}{\partial z_1} \\ \dfrac{\partial}{\partial z_2} \end{pmatrix} \qquad r^T = (z_1, z_2)$

Consequently:

$$\hat{a}_1 = -\frac{1}{2} i \left(z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2} \right) \qquad a_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\hat{a}_2 = -\frac{1}{2} \left(z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2} \right) \qquad a_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\hat{a}_3 = -\frac{1}{2} i \left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \right) \qquad a_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

If we define now $\Phi(a) = \hat{a} \ \forall a \in su(2)$ the operators corresponding to A_+, A_-, A_3 are

$$A_+ = -i \hat{a}_1 + i(-i \hat{a}_2) = -z_2 \frac{\partial}{\partial z_1}$$

$$A_- = -i \hat{a}_1 - i(-i \hat{a}_2) = -z_1 \frac{\partial}{\partial z_2}$$

$$A_3 = -i \hat{a}_3 = -\frac{1}{2} \left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \right)$$

From which we obtain

$$A_+ \psi^j_m(r) = (-1)^{m+1} \{(j-m)!\,(j+m)!\}^{-1/2} (j-m) z_1^{j-m-1} z_2^{j+m+1}$$

$$= (-1)^{m+1} \{(j-m-1)!\,(j+m+1)!\}^{-1/2} \{(j-m)(j+m+1)\}^{1/2} z_1^{j-(m+1)} z_2^{j+m+1}$$

$$= \{(j-m)(j+m+1)\}^{1/2} \psi^j_{m+1}(r)$$

$$A_- \psi^j_m(r) = (-1)^{m+1} \{(j-m)!\,(j+m)!\}^{-1/2} (j+m) z_1^{j-m+1} z_2^{j+m-1}$$

$$= (-1)^{m-1} \{(j-m+1)!\,(j+m-1)!\}^{-1/2} \{(j+m)(j-m+1)\}^{1/2} z_1^{j-(m-1)} z_2^{j+(m-1)}$$

$$= \{(j+m)(j-m+1)\}^{1/2} \psi^j_{m-1}(r)$$

Finally

$$A_3 \Psi^j_m = \frac{1}{2}(-1)^{m+1}\left[(j-m)-(j+m)\right]\{(j-m)!\,(j+m)!\}^{-1/2}\, z_1^{j-m} z_2^{j+m}$$

$$= m\,\Psi^j_m .$$

The basis functions for the representation $D^j_{\mathcal{G}}$ of SU(2) are also basis functions for the irreducible representation D^j of su(2). Consequently D^j is obtained from $D^j_{\mathcal{G}}$ by the limiting process

$$D^j(\hat{a}) = \left[\frac{d}{dt} D^j_{\mathcal{G}}(\exp(t\hat{a}))\right]_{t=0}$$

and consequently (since SU(2) is the covering group of itself)

$$D^j_{\mathcal{G}}(\exp t\hat{a}) = \exp(t\, D^j(\hat{a}))$$

expresses ALL possible elements of SU(2) if $\hat{a} \in su(2)$. The connection to the rotations can be obtained, for example via their quaternionic descriptions by noticing that:

$$\tfrac{1}{2}\mathbb{I} = \tfrac{1}{2}[\![0,\hat{i}]\!] \qquad \tfrac{1}{2}\mathbb{J} = \tfrac{1}{2}[\![0,\hat{j}]\!] \qquad \tfrac{1}{2}\mathbb{K} = \tfrac{1}{2}[\![0,\hat{k}]\!]$$

generate an algebra isomorphic to su(2) and that

$$R(\lambda,\Lambda) = [\![\cos\tfrac{\phi}{2},\ \sin\tfrac{\phi}{2}\,\vec{n}]\!] = \exp\left(\tfrac{\phi}{2}[\![0,\hat{n}]\!]\right)$$

is associated to the rotation of ϕ around $\vec{n}$. In particular, if $\hat{n}=\hat{k}=(0,0,1)$

$$R(\phi,\hat{k}) \leftrightarrow [\![\cos\tfrac{\phi}{2},\ \sin\tfrac{\phi}{2}\hat{k}]\!]$$

It follows that the character of the $2j+1$-dimensional representation of the element of SU(2) associated to the rotation of ϕ is

$$\chi^j(\phi) = \mathrm{Tr}\ \exp\left(i\phi\, D^j(A_3)\right) = \mathrm{Tr}\ \exp\left(\phi\, D^j(\hat{a}_3)\right)$$

$$= \sum_{m=-j}^{j} \exp i m \phi \underset{\text{see page 61.}}{=} \frac{\sin\left(j+\frac{1}{2}\right)\phi}{\sin\frac{\phi}{2}}$$

j is now, though semiinteger. This is the rigorous proof of the formula above that we have "guessed" at page 67 of these notes.

Now we turn to the representation of $SO(3)$, the general theory of the universal covering group ensures us that the representation D^j of $SU(2)$ is also a representation for $SO(3)$ if and only if $D^j(u) = \mathbb{1}_{2j+1}$ $\forall$ $u \in K$ the central invariant subgroup of $SU(2)$. We have already demonstrated that $K = \{\mathbb{1}_2, -\mathbb{1}_2\}$. Since D^j is an homomorphism $\Rightarrow$ $D^j(\mathbb{1}_2) = \mathbb{1}_{2j+1}$. We have only to look for j for which $D^j(-\mathbb{1}_2) = \mathbb{1}_{2j+1}$

$$a_3 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad \exp(\phi\, a_3) = -\mathbb{1}_2 \iff \phi = 2\pi$$

We can now consider the corresponding representation of dimension $2j+1$

$$D^j(\exp 2\pi\, \hat{a}_3) = \exp\left(2\pi\, D^j(\hat{a}_3)\right) = D^j(\widehat{-\mathbb{1}_2})$$

The matrix elements are:

$$\exp\left(2\pi\, D^j(\hat{a}_3)\right)_{mm'} = \delta_{mm'} \exp(i 2\pi m) = \delta_{mm'} \iff m \in \mathbb{Z}$$

We conclude that D^j is also a representation for $SO(3)$ if $j = 0, 1, 2, \ldots$

The equation containing only $\hat{L}_3$ is solved by $\psi^j_m(r) = e^{im\phi} f(r,\theta)$, which, substituted into the first of (**) gives:

$$\left\{ \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) - \frac{m^2}{\sin^2\theta} \right\} f(r,\theta) = j(j+1) f(r,\theta)$$

which is the associate Legendre equation with solution $f(r,\theta) = P^m_j(\cos\theta) F(r)$
With the appropriate normalization condition

$$\int dr\ \psi^j_m(r)\, \psi^{j'}_{m'}(r) = \delta_{jj'} \delta_{mm'}$$

$$\psi^j_m(r) = Y_{jm}(\theta,\phi) R(r)$$

where $\int_0^\infty dr\ r^2 |R(r)|^2 = 1$

and $Y_{jm}(\theta,\phi) = (-1)^m \left\{ \frac{(2j+1)}{4\pi} \frac{(j-m)!}{(j+m)!} \right\}^{1/2} e^{im\phi} P^m_j(\cos\theta)$

Finally we turn to O(3). O(3) is isomorphic to $SO(3) \otimes \{1_3, -1_3\}$.
The group $\{1_3, -1_3\}$ has only 2 irreducible representations, both 1 dimensional

$$\begin{cases} \Gamma^1(1_3) = 1 & \Gamma^1(-1_3) = +1 \\ \Gamma^{-1}(1_3) = 1 & \Gamma^{-1}(-1_3) = -1 \end{cases}$$

One deduces (using results from Ch. 9) that O(3) has only 2 inequivalent irreducible representations for $j = 0, 1, 2 \ldots$ which we denote with $\Gamma^{p,j}$

$$\Gamma^{1j}(R) = D^j(R) \qquad \Gamma^{1j}(-R) = D^j(R)$$
$$\Gamma^{-1j}(R) = D^j(R) \qquad \Gamma^{-1j}(-R) = -D^j(R)$$

Since $-\hat{1}_3\, Y_{jm}(\theta,\phi) = Y_{jm}(\pi-\theta, \varphi+\pi) = (-1)^j\, Y_{jm}(\theta,\varphi)$, the basis functions $Y_{jm}(\theta,\varphi) F(r)$ are basis functions for $\Gamma^{p,j}$ only when $p = (-1)^j$.

The representations $\Gamma^{p,j}$ with $p = -(-1)^j$ has no basis functions.
This does not imply, through that they have no physical meaning.
The most important example, perhaps is the one of the angular momentum operators L_x, L_y, L_z which are a three dimensional representation of $SO(3)$. Moreover they are even with respect to inversion
$\Rightarrow$ L_x, L_y, L_z transform as irreducible tensor operators of a representation of $O(3)$ equivalent to $\Gamma^{1,1}$.