Quantum theory of condensed matter II

Mesoscopic physics

Prof. Milena Grifoni	Tue	8:00 - 10:00	9.2.01
	Fri	10:00 - 12:00	H33
PD Dr. Andrea Donarini	Fri	12:00 - 14:00	5.0.20

Sheet 9

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1. Occupation number representation

Let us consider a fermionic system with two single particle states $|\alpha\rangle$ and $|\beta\rangle$ that span the (two-dimensional) *one*-particle Hilbert space.

1. What is the dimension of the *two*-particle Hilbert space? What is the dimension of the Fock space? Write down the basis of the Fock space explicitly as Slater determinants of the wave functions $\phi_\alpha(\mathbf{r}), \phi_\beta(\mathbf{r})$ and in the occupation number representation. **(2 Points)**
2. Calculate, in the Fock basis, the matrix representation of the creation and annihilation operators $\hat{c}_\mu, \hat{c}_\mu^\dagger$ $(\mu = \alpha, \beta)$ and also of the occupation operators $\hat{n}_\mu = \hat{c}_\mu^\dagger \hat{c}_\mu$. **(2 Points)**
3. Using explicitly the matrix multiplication of the matrices calculated in (b), calculate the anticommutator relations $[\hat{c}_\mu, \hat{c}_\nu]_+ = \left[\hat{c}_\mu^\dagger, \hat{c}_\nu^\dagger\right]_+ = 0$ and $\left[\hat{c}_\mu, \hat{c}_\nu^\dagger\right]_+ = \delta_{\mu\nu}$. **(2 Points)**

2. Wick's theorem

1. Show that, for a system of non-interacting fermions described by the Hamiltonian $\hat{H} = \sum_\alpha \epsilon_\alpha \hat{c}_\alpha^\dagger \hat{c}_\alpha$, the following relation for the many-body grandcanonical expectation value holds:
$$\langle \hat{c}_{\alpha_1}^\dagger \hat{c}_{\alpha_2}^\dagger \hat{c}_{\alpha_3} \hat{c}_{\alpha_4} \rangle = \langle \hat{c}_{\alpha_1}^\dagger \hat{c}_{\alpha_4} \rangle \langle \hat{c}_{\alpha_2}^\dagger \hat{c}_{\alpha_3} \rangle \delta_{\alpha_1\alpha_4}\, \delta_{\alpha_2\alpha_3} - \langle \hat{c}_{\alpha_1}^\dagger \hat{c}_{\alpha_3} \rangle \langle \hat{c}_{\alpha_2}^\dagger \hat{c}_{\alpha_4} \rangle \delta_{\alpha_1\alpha_3}\, \delta_{\alpha_2\alpha_4},$$
where
$$\langle \hat{c}_{\alpha_1}^\dagger \hat{c}_{\alpha_2}^\dagger \hat{c}_{\alpha_3} \hat{c}_{\alpha_4} \rangle \equiv \frac{1}{Z} \operatorname{Tr} \left\{ \hat{c}_{\alpha_1}^\dagger \hat{c}_{\alpha_2}^\dagger \hat{c}_{\alpha_3} \hat{c}_{\alpha_4} \, \exp\left[-\beta(H - \mu N)\right] \right\}$$
and Z is the grandcanonical partition function. The trace is taken over the full Fock space. **(2 Points)**
2. Derive from 2.1 that, for noninteracting fermions, in every other given single particle basis $\{|n\rangle\}$ the following relation holds:
$$\langle \hat{c}_{n_1}^\dagger \hat{c}_{n_2}^\dagger \hat{c}_{n_3} \hat{c}_{n_4} \rangle = \langle \hat{c}_{n_1}^\dagger \hat{c}_{n_4} \rangle \langle \hat{c}_{n_2}^\dagger \hat{c}_{n_3} \rangle - \langle \hat{c}_{n_1}^\dagger \hat{c}_{n_3} \rangle \langle \hat{c}_{n_2}^\dagger \hat{c}_{n_4} \rangle.$$
Note that this is valid even if in this basis the Hamiltonian
$$\hat{H} = \sum_{n,m} h_{nm} \hat{c}_n^\dagger \hat{c}_m$$
would contain non-diagonal terms, h_{nm} for $n \neq m$. Hint: Diagonalize H first, using a unitary transformation $\hat{c}_n = \sum_\alpha u_{n\alpha} \hat{c}_\alpha$. Apply the equation proven in 2.1. Finally perform the canonical transformation in the reverse direction. **(2 Points)**

Frohe Weihnachten!