University of Regensburg | Winter Term 16/17

Quantum theory of condensed matter II

Mesoscopic physics

Prof. Milena Grifoni	Tue	8:00 - 10:00	9.2.01
	Fri	10:00 - 12:00	H33
PD Dr. Andrea Donarini	Fri	12:00 - 14:00	5.0.20

Sheet 8

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1. Calculating with bosonic operators

Refresh the physics of the simple harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2},$$

which can be written in "second quantized" form, by expressing $\hat{x}$ and $\hat{p}$ in terms of *boson* creation and annihilation operators:

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \quad a^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\hat{x} - \mathrm{i}\frac{\hat{p}}{\sqrt{m\omega}}\right).$$

From the canonical commutation relations between position and momentum operators, it follows immediately (do you remember it?) that the basic commutation relations hold:

$$[a, a^\dagger] = 1, \quad [a, a] = 0,$$

where $[A, B] = AB - BA$, $|0\rangle$ is the vacuum, and $\dagger$ indicates the Hilbert space adjoint.

1. Show that for two non commuting operators A, and B it holds

$$[A, B^n] = \sum_{k=0}^{n-1} B^k\,[A, B]\,B^{n-1-k}.$$

2. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$ and prove the following relation:

$$[b, f(b^\dagger)] = f'(b^\dagger),$$

where $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$ and b is a bosonic operator.
Hint: You can start by proving, with the help of 1.1, that

$$\left[b, (b^\dagger)^n\right] = n(b^\dagger)^{n-1}.$$

(3 Points)

2. Exponential of bosonic operators

A particular role is played in quantum mechanics by exponential operators. Time evolution, spatial translation and any transformation associated to a continuum symmetry group is represented by an exponential operator. Thus we dedicate a special exercise to them.

1. Using the previous arguments (Ex. 1.2) show that the following relation hold

$$g_1(\alpha; b, b^\dagger) = \mathrm{e}^{-\alpha b^\dagger} b \mathrm{e}^{\alpha b^\dagger} = b + \alpha$$

2. Simplify the following expression

$$g_2(\alpha; b, b^\dagger) = \mathrm{e}^{-(\alpha^* b^\dagger - \alpha b)} b \mathrm{e}^{(\alpha^* b^\dagger - \alpha b)}.$$

Hint: Introduce a "dummy" variable λ, consider the auxiliary function:

$$\tilde{g}_2(\lambda, \alpha; b, b^\dagger) = \mathrm{e}^{-\lambda(\alpha^* b^\dagger - \alpha b)} b \mathrm{e}^{\lambda(\alpha^* b^\dagger - \alpha b)}$$

and calculate the derivative $\partial \tilde{g}_2(\lambda, \alpha; b, b^\dagger)/\partial\lambda$. Notice that:

$$\begin{aligned} \tilde{g}_2(1, \alpha; b, b^\dagger) &= g_2(\alpha; b, b^\dagger) \\ \tilde{g}_2(0, \alpha; b, b^\dagger) &= b. \end{aligned}$$

(2 Points)

3. Calculating with fermionic operators

The basis commutation relations for fermion creation and annihilation operators are

$$[c, c^\dagger]_+ = 1, \quad [c, c]_+ = 0, \quad c|0\rangle = 0,$$

where $[A, B]_+ = AB + BA$, $|0\rangle$ the vacuum, and $\dagger$ indicates the Hilbert space adjoint. Similarly to exercise 2, simplify the following expressions involving, this time, the fermionic operators c, and $c^\dagger$

$$\begin{aligned} g(\alpha; c, c^\dagger) &= \mathrm{e}^{\alpha c^\dagger} c \mathrm{e}^{\alpha c^\dagger}, \\ h(\alpha; c, c^\dagger) &= \mathrm{e}^{-\alpha c^\dagger c} c \mathrm{e}^{\alpha c^\dagger c}. \end{aligned}$$

(2 Points)

Frohes Schaffen!