Quantum Theory of Condensed Matter II

Mesoscopic Physics

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Sheet 3

1. Scattering on impurities

One of the most delicate issues in the formulation of the Boltzmann equation is the calculation of the collision integral. Let us consider the case of non interacting electrons scattered by impurities.

1. Prove that, assuming local elastic scattering on the impurities, the collision integral can be written in the form:

$$\begin{aligned}\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} &= \frac{V}{(2\pi\hbar)^3}\int \mathrm{d}\mathbf{p}'\frac{2\pi}{\hbar}|\langle\mathbf{p}|\hat{V}_{\text{imp}}|\mathbf{p}'\rangle|^2\delta(\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{p}'})[f(\mathbf{r},\mathbf{p}',t)-f(\mathbf{r},\mathbf{p},t)] \\ &= \frac{V}{(2\pi\hbar)^3}\int \mathrm{d}\mathbf{p}' W_{\mathbf{p}\mathbf{p}'}[f(\mathbf{r},\mathbf{p}',t)-f(\mathbf{r},\mathbf{p},t)]\end{aligned}$$

   where $\hat{V}_{\text{imp}} = \sum_{i=1}^{N_i} V_0(\hat{r}-\mathbf{R}_i)$ is the potential (operator) generated by the impurities located at positions $\mathbf{R}_i$ and V is the volume of the metal.

2. Prove that, for a random distribution of the impurity positions $\mathbf{R}_i$, the scattering rate $W_{\mathbf{p}\mathbf{p}'}$ is proportional to the impurity density $n_i = N_i/V$. In addition, prove that, for free electrons, and s-scatterers (*i.e.* $V_0(\mathbf{r}) = V_0(|\mathbf{r}|)$) one obtains the following dependance of the scattering rate on the incoming and outgoing wave vectors: $W_{\mathbf{p}\mathbf{p}'} = W\left(|\mathbf{p}|, \mathbf{e}_{\mathbf{p}}\cdot\mathbf{e}_{\mathbf{p}'}\right)$, where $\mathbf{e}_{\mathbf{v}} \equiv \mathbf{v}/|\mathbf{v}|$.

3. Calculate explicitly the momentum relaxation time $\tau_{\text{tr}}(|\mathbf{p}|)$, using an impurity potential of the form

$$V_0\left(\mathbf{r}-\mathbf{R}_i\right) = \begin{cases} -U, & |\mathbf{r}-\mathbf{R}_i| < a \\ 0, & |\mathbf{r}-\mathbf{R}_i| \geq a\end{cases}$$

   and the definition of $\tau_{\text{tr}}(|\mathbf{p}|)$:

$$\tau_{\text{tr}}(|\mathbf{p}|)^{-1} = \frac{V}{(2\pi\hbar)^3}\int \mathrm{d}\mathbf{p}' W_{\mathbf{p}\mathbf{p}'}\left(1-\frac{\mathbf{p}\cdot\mathbf{p}'}{p^2}\right)$$

   Discuss the limit $ak_{\text{F}} \ll 1$.

(5 Points)

2. Dynamical conductivity in the relaxation time model

In this exercise you will calculate, by means of the Boltzmann equation, the conductivity σ and the density correlator χ of a simple metal for arbitrary frequencies ω and wave numbers $\mathbf{k}$ in 3 spatial dimensions. Assume that the scattering centers are isotropic scatterers, *i.e.* $W_{\mathbf{p}\mathbf{p}'}$ is independent of the angles of the incident and scattered momenta $\mathbf{p}$ and $\mathbf{p}'$.

1. First show that, in the isotropic case, the collision term reduces to:

$$\left(\frac{df}{dt}\right)_{\rm coll} = -\frac{1}{\tau}(f - f_l)$$

where

$$f_l(\mathbf{r}, |\mathbf{p}|, t) = \int \frac{d\Omega_{\mathbf{p}}}{4\pi} \; f(\mathbf{r}, \mathbf{p}, t).$$

2. As a second step, linearize the Boltzmann equation around the local equilibrium distribution and perform on it a time and space Fourier transformation. Assume the force term originating from a longitudinal electric field $\mathbf{E}(\mathbf{q}, \omega) = -i\mathbf{q}\phi(\mathbf{q}, \omega)$. Derive then a closed expression for δf_l in terms of the Lindhardt function

$$\mathcal{L} = \int \frac{i\mathbf{v}\cdot\mathbf{q}}{1/\tau - i\omega + i\mathbf{v}\cdot\mathbf{q}} \frac{d\Omega_{\mathbf{p}}}{4\pi},$$

where $\mathbf{v} = \mathbf{p}/m$.

3. The variation of the particle density $\delta n(\mathbf{q}, \omega)$ (obtained by integrating over momenta the variation of the Boltzmann distribution function) is proportional to the applied potential $-e\phi(\mathbf{q}, \omega)$. Which is the form of the proportionality factor $\chi(q, \omega)$? Moreover, show that the relation between χ and the longitudinal conductivity σ_l reads:

$$\sigma_l = e^2 \frac{-i\omega}{q^2} \chi.$$

4. Discuss the asymptotic behaviour of σ_l. Calculate for this purpose $\mathcal{L}$ in the limits $q\ell \ll |1 - i\omega\tau|$ and $q\ell \gg |1 - i\omega\tau|$. ($\ell$ is the mean free path $v_{\rm F}\tau$)

5. Finally, consider a transversal electric field. Prove that, in this case, it holds $\delta f_l = 0$. Calculate the transversal conductivity in the asymptotic limits. How do you explain the differences with respect to the longitudinal response?

(10 Points)

Frohes Schaffen!