Universität Regensburg Winter Term 14/15

Quantum Theory of Condensed Matter II

Mesoscopic Physics

Prof. Ferdinand Evers
Dr. Andrea Donarini

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Sheet 2

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1. Density of states

The density of states is a basic quantity useful to characterize a physical system. Fingerprints of the shape of the density of states are observed both in the equilibrium state and non-equilibrium dynamics.

1. For an electron gas in zero (quantum dot), one (quantum wire), two (2DEG), and three (bulk) dimensions, calculate the density of states $\rho(E)$ making use of the effective mass approach. Sketch the function $\rho(E)$ taking into account the level quantization for the low dimensional systems.
2. For each of the cases calculate also the compressibility $\frac{\mathrm{d}n}{\mathrm{d}\mu}$ where n is the particle density and μ the chemical potential (*i.e.*, at $T = 0$, the Fermi energy) for the system.
3. What changes if the electrons follow a linear dispersion relation?

(4 Points)

2. Einstein's relation

The classical Drude formula $\sigma = \frac{e^2 n \tau_{\mathrm{tr}}}{m}$ expresses the conductivity of a system in terms of the electric charge e, the (effective) mass m, the density n of the charge carriers and their momentum relaxation time τ_m.

1. Derive the classical Drude formula starting from the the equation of motion for the (average) momentum in presence of momentum relaxation and of an external electric field: $\dot{\mathbf{p}} = -\mathbf{p}/\tau_{\mathrm{tr}} - e\mathbf{E}$.
2. The diffusion coefficient D is connected to the Fermi velocity v_{F} and the momentum relaxation time τ_{tr}. via the relation $D = v_{\mathrm{F}}^2 \tau_{\mathrm{tr}}/d$ where d is the dimensionality of the system. Calculate the conductivity according to the Einstein relation $\sigma = e^2 D \rho(E_{\mathrm{F}})$ where $\rho(E_{\mathrm{F}})$ is the density of states at the Fermi energy and compare it with the result obtained from the classical Drude formula. Do the calculation for one, two and three dimensions.
3. Establish that the Einstein's relation and the classical Drude formula are equivalent only under the condition $\frac{v_{\mathrm{F}^2}}{d} \frac{dn}{d\mu}|_{\mu=E_F} = \frac{n}{m}$. Consequently, discuss the generality of the result.

(4 Points)

3. Liouville's theorem

Consider a set of N identical independent particles classically described by their positions $(\mathbf{q}_i)$ and momenta $(\mathbf{p}_i)$. Their dynamics is governed by the Hamiltonian $H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V(\mathbf{q}_i)$.

1. Let us now introduce the time dependent distribution $\mathcal{G}(\mathbf{q}, \mathbf{p}, t)$ defined on the 6-dimensional configuration space:
$$\mathcal{G}(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^{N} \delta(\mathbf{q} - \mathbf{q}_i(t))\delta(\mathbf{p} - \mathbf{p}_i(t)),$$
where δ is the Dirac delta distribution function. Prove that the distribution $\mathcal{G}$ satisfies the equation of motion:

$$\frac{\partial \mathcal{G}}{\partial t} + \frac{1}{m}\mathbf{p} \cdot \nabla_{\mathbf{q}}\mathcal{G} + \mathbf{F} \cdot \nabla_{\mathbf{p}}\mathcal{G} = 0,$$

where we have introduced the force $\mathbf{F} \equiv -\nabla_{\mathbf{q}} V$. Compare the result with the Boltzmann equation derived in class.

2. Verify that the integral of $\mathcal{G}$ over the entire configuration space is conserved *i.e.*:

$$N = \int \mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p}\, \mathcal{G}(\mathbf{q}, \mathbf{p}, t), \qquad \forall t.$$

3. The conservation of the number of particles is not only a global property of $\mathcal{G}$. Prove that also the following relation holds:

$$\mathcal{G}(\mathbf{q}, \mathbf{p}, t)\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p} = \mathcal{G}(\mathbf{q}', \mathbf{p}', t')\mathrm{d}\mathbf{q}'\mathrm{d}\mathbf{p}'$$

where $\mathbf{q}'$ and $\mathbf{p}'$ are the coordinate and momentum at time t' of a particle that had coordinate and momentum $\mathbf{q}$ and $\mathbf{p}$ at time t and is described by the Hamiltonian $H = p^2/2m + V(\mathbf{q})$.
Hint: The most difficult part is the proof of the differential volume conservation in d dimensions. Verify that the determinant of the Jacobian corresponding to the infinitesimal transformation $\bar{p}' = \bar{p} + \dot{\bar{p}}\,\mathrm{d}t$ and $\bar{q}' = \bar{q} + \dot{\bar{q}}\,\mathrm{d}t$ equals 1 up to first order in $\mathrm{d}t$.

(4 Points)

Frohes Schaffen!