Mesoscopic Physics

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Room 3.1.26
Fridays at 10:15

Sheet 9

1. Combination of S-matrices and the double-δ potential

Consider a quasi-1D wire (only 1 propagating mode) with two identical scatterers scatterers at $x = 0$ and $x = d$ respectively. The scattering potential is approximated by

$$U(x) = U_0 \left[\delta(x) + \delta(x - d)\right] \ . \tag{1}$$

(a) Show that the transmission and reflection reflection probabilities are given by

$$T_1 = \frac{\hbar^2 v^2}{\hbar^2 v^2 + U_0^2} \quad R_1 = \frac{U_0^2}{\hbar^2 v^2 + U_0^2} \tag{2}$$

with the velocity $v = \sqrt{2E/m}$.
(*Hint:* Remember the special matching conditions for the wavefunctions at a $\delta-$like potential from QM I.)

(b) Use the procedure of coherent S-matrix combination to show that the total transmission probability is given by

$$T = T(E) = \frac{T_1^2}{1 - 2R_1 \cos(\theta) + R_1^2} \tag{3}$$

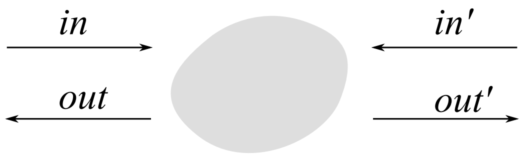
with $\theta = 2\left[dmv/\hbar + \tan^{-1}(\hbar v/U_0)\right]$ and plot $T(E)$ for $U_0 = 9\,\text{eV}\,\text{Å}$, $d = 50\,\text{Å}$ and $0 < E < 250\,\text{meV}$.

(c) • Resonant transmission: Although the individual transmission probability T_1 is usually very small, T can become large for certain resonant energies. What is the maximum value for T? For strong scatterers $U_0 \gg \hbar v$, calculate the positions of the resonances E_n.

please turn over

2. Transfer matrix formalism and the double barrier potential

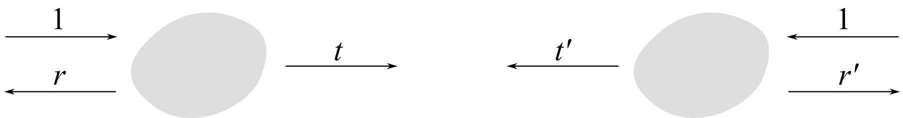
While the scattering matrix S connects incoming and outgoing waves, the *transfer matrix* $\mathcal{T}$ connects waves on the left side and waves on the right side of a scatterer in a (quasi-)onedimensional problem.



In the schematic notation of the figure, the transfer matrix is defined as

$$\begin{pmatrix} out' \\ in' \end{pmatrix} = \mathcal{T} \begin{pmatrix} in \\ out \end{pmatrix} \tag{4}$$

(a) • Show that for a scatterer with transmission and reflection amplitudes t and r for particles coming from the left and t' and r' for particles coming from the right



the transfer matrix is

$$\mathcal{T} = \begin{pmatrix} t - \frac{rr'}{t'} & \frac{r'}{t'} \\ & \\ -\frac{r}{t'} & \frac{1}{t'} \end{pmatrix} \tag{5}$$

By convention within the transfer matrix formalism, the waves are normalized to unit probability, not to unit flux as for the scattering matrix. Therefore the Transmission is in general given by

$$T = \frac{v'}{v}|t|^2 = \frac{v}{v'}|t'|^2 = \frac{v}{v'}\frac{1}{|\mathcal{T}_{22}|^2}\,, \tag{6}$$

where v and v' are the velocities on the left and right side respectively.

(b) • If one has a series of scatteres, how are the individual transfer matrices combined to give the transfer matrix of the full system?

(c) Consider a rectangular barrier

$$U(x) = \begin{cases} U & a \le x \le b \\ 0 & \text{else} \end{cases} \tag{7}$$

and use the transfer matrix formalism to show that the transmission and reflection amplitudes for $E < U$ are given by

$$t = \frac{e^{ik(b-a)}}{\cosh[\kappa(b-a)] + i\frac{\varepsilon}{2}\sinh[\kappa(b-a)]} \qquad r = -i\frac{\eta}{2}\frac{\sinh[\kappa(b-a)]\,e^{ik(b-a)}}{\cosh[\kappa(b-a)] + i\frac{\varepsilon}{2}\sinh[\kappa(b-a)]} \tag{8}$$

with $\kappa = \sqrt{2m(U-E)}/\hbar$, $\varepsilon = \kappa/k - k/\kappa$ and $\eta = \kappa/k + k/\kappa$. Herefore determine the transfer matrices for the first potential step, the piece within the barrier and the second step and combine them.

(d) • Now consider two barriers in series with transfer matrices $\mathcal{T}_1$ and $\mathcal{T}_2$ and transmission/reflection amplitudes t_1, r_1, t_2 and r_2.

$$U(x) = \begin{cases} U_1 & 0 \le x \le W_1 \\ U_2 & W_1 + d \le x \le W_1 + d + W_2 \\ 0 & \text{else} \end{cases} \tag{9}$$

Note that the free space between the barriers also has a transfer matrix $\mathcal{T}_d$. Combine the transfer matrices again to show that the total Transmission is given by

$$T = \left| \frac{t_1 t_2}{1 - r_1 r_2 e^{2ikd}} \right|^2 \tag{10}$$

(*Hint:* Use the relation $(\mathcal{T}_{1/2})_{11} = (\mathcal{T}_{1/2})^*_{22}$.
Can you prove by it looking at the time reversed problem $\psi \to \psi^*$?)

(e) • Plot T for identical barriers ($U_1 = U_2$, $W_1 = W_2$) for different parameter sets such that

$$\begin{aligned} d + \frac{1}{2}(W_1 + W_2) &= 50\,\text{Å} \\ W_1\,U_1 = W_2\,U_2 &= 9\,\text{eV} \\ U_1, U_2 &< 250\,\text{meV} \\ 0 < E &< 250\,\text{meV}\,. \end{aligned}$$

Compare with exercise 1.

(f) Use the method of finite differences and the Fisher-Lee relations to solve the double barrier problem numerically:

- Show that the second derivative of the wavefunction has to be discretized as

$$\begin{aligned} \psi(x) &\to \psi(x_i) \equiv \psi_i \\ \psi''(x) \to \psi''(x_i) &= \frac{\psi_{i+1} - 2\psi_i + \psi_{i-1}}{a^2} \end{aligned} \tag{11}$$

  with the lattice spacing a. This leads to the discrete Schrödinger equation $H_{ij}\psi_j = E\psi_i$ with

$$H_{ij} = (2t + V_i)\delta_{ij} - t\delta_{i+1,j} - t\delta_{i-1,j}\,. \tag{12}$$

  $t = \hbar^2/2ma^2$ is the hopping parameter and $V_i \equiv V(x_i)$.
- In the lecture it was shown that the problem of inverting the full (infinite) matrix $E - H$ can be avoided by using the finite sized Hamiltonian of the scattering region H_S and taking the leads into account by adding the so-called self energy $\Sigma^{R/A}$ with

$$\Sigma^R_{ij} = -\delta_{ij} t e^{ika} \left[\delta_{1j} + \delta_{Nj}\right] \tag{13}$$

  and $\Sigma^A_{ij} = (\Sigma^A_{ji})^*$ for identical leads in 1D. "1" and "N" are the first and the last point in the scattering region respectively. The retarded/advanced Green function of the scattering region is then

$$G^{R/A}_S = \frac{\hbar}{a}\left(E - H_S - \Sigma^{R/A}\right)^{-1}\,. \tag{14}$$

  Set up H_S and calculate $G^{R/A}_S$ numerically. You can use for example Matlab to invert the matrix.
- Relate the Green function to the transmission using the Fisher-Lee relation derived in class

$$T = \text{Tr}\left[\Gamma\, G^R_S\, \Gamma\, G^A_S\right] \qquad \Gamma = i[\Sigma^R - \Sigma^A]\,. \tag{15}$$

- Plot the result for different parameters and lattice spacings.

Frohes Schaffen!