Equation of motion

Given $A(\tau)$ the imaginary time evolution of A:

$$\partial_\tau A(\tau) = \partial_\tau\left(e^{\tau H} A\, e^{-\tau H}\right) = [H, A](\tau)$$

$\Rightarrow$ the τ derivative of the Matsubara correlator:

$$-\partial_\tau C_{AB}(\tau-\tau') = \delta(\tau-\tau')\langle AB - (\pm)BA\rangle + \langle T_\tau\{[H,A](\tau)\, B(\tau')\}\rangle$$

Thus for the Matsubara single particle Green's functions:

$$-\partial_\tau \mathcal{G}(r\tau, r'\tau') = \delta(\tau-\tau')\delta(r-r') + \langle T_\tau\{[H,\psi(r)](\tau)\,\psi^+(r',\tau')\}\rangle$$

$$-\partial_\tau \mathcal{G}(\nu\tau, \nu'\tau') = \delta(\tau-\tau')\delta_{\nu\nu'} + \langle T_\tau\{[H, c_\nu](\tau)\, c^+_{\nu'}(\tau')\}\rangle$$

For non-interacting hamiltonians

$$-\partial_\tau \mathcal{G}_0(r\tau, r'\tau') - \int dr''\, h_0(r,r'')\, \mathcal{G}_0(r''\tau, r'\tau') = \delta(\tau-\tau')\delta(r-r') \quad \text{position representation}$$

$$-\partial_\tau \mathcal{G}_0(\nu\tau, \nu'\tau') - \sum_{\nu''} h_{0,\nu\nu''}\, \mathcal{G}_0(\nu''\tau, \nu'\tau) = \delta(\tau-\tau')\,\delta_{\nu\nu'} \quad \text{in } \nu \text{ representation}$$

example

$$h_{0,\nu\nu'} = \delta_{\nu\nu'}\varepsilon_\nu \Rightarrow -\partial_\tau \mathcal{G}_0(\nu\tau,\nu'\tau') - \varepsilon_\nu \mathcal{G}_0(\nu\tau,\nu'\tau') = \delta(\tau-\tau')\delta_{\nu\nu'}$$

In the frequency domain $\mathcal{G}_0(\nu, i\omega_n) = \dfrac{1}{i\omega_n - \varepsilon_\nu} \sim \delta_{\nu\nu'} \ldots$

In other terms. $\mathcal{G}_0^{-1} = -\partial_\tau - h_0$

Wick's theorem

For non-interacting particles, higher order Green's functions can be factorized into products of single particle Green's functions

$$\mathcal{G}_0^{(n)}(1,\ldots,n;1',\ldots n') = \begin{vmatrix} \mathcal{G}_0(1,1') & \cdots & \mathcal{G}_0(1,n') \\ \vdots & \ddots & \vdots \\ \mathcal{G}_0(n,1') & \cdots & \mathcal{G}_0(n,n') \end{vmatrix}_{B,F}, \quad i \equiv (\nu_i, \tau_i)$$

$||_{B,F}$ means permanent or determinant for bosons or fermions respectively.

<u>PROOF</u>

SEE ALSO 178-b, 178-C

$$\mathcal{G}_0^{(n)}(\nu_1\tau_1,\dots,\nu_n\tau_n;\nu_1'\tau_1',\dots,\nu_n'\tau_n') =$$

$$= (-1)^n \left\langle T_\tau\left[c_{\nu_1}(\tau_1)\cdots c_{\nu_n}(\tau_n)c^+_{\nu_n'}(\tau_n')\cdots c^+_{\nu_1'}(\tau_1')\right]\right\rangle_0$$

where imaginary time evolution and thermal average are performed with respect to a non-interacting Hamiltonian.

notation
$$d_j(\sigma_j) = \begin{cases} c_{\nu_j}(\tau_j) & j\in[1,n] \\ c^+_{\nu'(2n+1-j)} & j\in[n+1,2n]\end{cases}$$

$$P\left(d_1(\sigma_1)\cdots d_{2n}(\sigma_{2n})\right) = d_{P_1}(\sigma_{P_1})\cdots d_{P_{2n}}(\sigma_{P_{2n}})$$

$$\Rightarrow \quad \mathcal{G}_0^{(n)}(j_1\cdots j_{2n}) = (-1)^n\sum_{P\in S_{2n}}(\pm1)^P\,\theta(\sigma_{P_1}-\sigma_{P_2})\cdots\theta(\sigma_{P_{n-1}}-\sigma_{P_n})$$
$$\times\langle d_{P_1}(\sigma_{P_1})\cdots d_{P_{2n}}(\sigma_{P_{2n}})\rangle_0$$

Notice that only one element in the sum satisfy all the theta functions. Using the equation of motion and the formal distinction $\partial_\tau = \partial_\tau^\theta + \partial_\tau^{\text{non-}\theta}$

$$\mathcal{G}_{0,i}^{-1}\mathcal{G}_0^{(n)} = -\partial^\theta_{\tau_i}\mathcal{G}_0^{(n)}$$

Now let's assume τ_i next to τ_j' (one is associated to a c, the other to a c^+ operator).

$$\mathcal{G}_0^{(n)} = [\dots\theta(\tau_i-\tau_j')\dots]\langle\dots c_{\nu_i}(\tau_i)c^+_{\nu_j'}(\tau_j')\dots\rangle$$
$$\pm[\dots\theta(\tau_j'-\tau_i)\dots]\langle\dots c^+_{\nu_j'}(\tau_j')c_{\nu_i}(\tau_i)\dots\rangle$$

$$\rightarrow\quad -\partial^\theta_{\tau_i}\mathcal{G}_0^{(n)} = \left([\dots]\langle\dots c_{\nu_i}(\tau_i)c^+_{\nu_j'}(\tau_j')\dots\rangle \mp[\dots]\langle\dots c^+_{\nu_j'}(\tau_j')c_{\nu_i}(\tau_i)\dots\rangle\right)\delta(\tau_i-\tau_j')$$

$$\left[c_{\nu_i}(\tau_i), c^+_{\nu_j'}(\tau_i)\right]_{B,F} = \delta_{\nu_i\nu_j'}.$$

In case τ_i is next to τ_j $\Rightarrow$ $\partial^\theta_{\tau_i}\mathcal{G}_0^{(n)} = 0$

$$\mathcal{G}_{0,i}^{-1}\mathcal{G}_0^{(n)} = \sum_{j=1}^n\delta_{\nu_i\nu_j}\,\delta(\tau_i-\tau_j')(-1)^x\,\mathcal{G}_0^{(n-1)}(\underbrace{\nu_1\tau_1\cdots\nu_n\tau_n}_{\text{without } i};\underbrace{\nu_1'\tau_1'\cdots\nu_n'\tau_n'}_{\text{without } j})$$

$(-1)^x$ depends on, def of $\mathcal{G}_0^{(n)}$, $\mathcal{G}_0^{(n-1)}$; operator permutations to bring c_{ν_i} close to $c^+_{\nu_j'}$

Wick's theorem at finite temperature

$$\mathcal{G}_0^{(n)}(1,\dots,n;1',\dots,n') = \begin{vmatrix} \mathcal{G}_0^{(1)}(1;1') & \dots & \mathcal{G}_0^{(1)}(1;n') \\ \vdots & \ddots & \vdots \\ \mathcal{G}_0^{(1)}(n;1') & \dots & \mathcal{G}_0^{(1)}(n;n') \end{vmatrix}_{B,F} \qquad i \equiv (\nu_i;\tau_i)$$

$$\mathcal{G}_0^{(n)}(\nu_1\tau_1,\dots,\nu_n\tau_n) \equiv (-1)^n \langle T_\tau\left[c_{\nu_1}(\tau_1)\dots c_{\nu_n}(\tau_n)\, c^\dagger_{\nu_n'}(\tau_n')\dots c^\dagger_{\nu_1'}(\tau_1')\right]\rangle_0$$

The Greens function can always be written as sum of θ function corresponding to al possible orderings. e.g.

$$\mathcal{G}_0^{(1)}(\nu_1\tau_1;\nu_1'\tau_1') \equiv -\langle T_\tau\left[c_{\nu_1}(\tau_1)\, c^\dagger_{\nu_1'}(\tau_1')\right]\rangle_0 =$$

$$= -\theta(\tau_1-\tau_1')\langle c_{\nu_1}(\tau_1)\, c^\dagger_{\nu_1'}(\tau_1')\rangle_0 - (\pm 1)\,\theta(\tau_1'-\tau_1)\langle c^\dagger_{\nu_1'}(\tau_1')\, c_{\nu_1}(\tau_1)\rangle_0$$

In particular it is usefull the relation derived from here

$$H \equiv \sum_{\nu_1\nu_2} c^\dagger_{\nu_1} c_{\nu_2} h_{\nu_1\nu_2} \quad \Rightarrow \quad \partial_\tau c_\nu(\tau) = -\sum_\mu h_{\nu\mu}\, c_\mu(\tau)$$

$$\partial_{\tau_1}\mathcal{G}_0^{(1)}(\nu_1\tau_1;\nu_1'\tau_1') = -\delta(\tau_1-\tau_1')\langle\left[c_{\nu_1}(\tau_1), c^\dagger_{\nu_1'}(\tau_1')\right]_{B,F}\rangle_0$$

$$-\theta(\tau_1-\tau_1')\langle \partial c_{\nu_1}(\tau_1)\, c^\dagger_{\nu_1'}(\tau_1')\rangle_0 - (\pm 1)\,\theta(\tau_1'-\tau_1)\langle c^\dagger_{\nu_1'}(\tau_1')\,\partial_{\tau_1} c_{\nu_1}(\tau_1)\rangle_0$$

$$= -\delta(\tau_1-\tau_1')\,\delta_{\nu_1\nu_1'} + \sum_\mu h_{\nu_1\mu}\,\mathcal{G}_0^{(1)}(\mu\tau_1;\nu_1'\tau_1')$$

$$\Rightarrow \quad \delta(\tau_1-\tau_1')\,\delta_{\nu_1\nu_1'} = -\sum_\mu\left(\partial_{\tau_1}\delta_{\nu_1\mu} + h_{\nu_1\mu}\right)\mathcal{G}_0^{(1)}(\mu\tau_1;\nu_1'\tau_1')$$

$$\partial_{\tau_i}\mathcal{G}_0^{(n)}(\nu_1\tau_1,\dots,\nu_n\tau_n;\nu_1'\tau_1',\dots,\nu_n'\tau_n') = \partial^\theta_{\tau_i}\mathcal{G}_0^{(n)} + \partial^{\text{non-}\theta}_{\tau_i}\mathcal{G}_0^{(n)}$$

$$\partial^\theta_{\tau_i}\mathcal{G}_0^{(n)} = (-1)^n\sum_{j\,P_{ij}} (\pm 1)^{n-j+n-i}\,\delta(\tau_i-\tau_j')\,\left[\dots\right]_\theta \langle\dots\left[c_{\nu_i}(\tau_i), c^\dagger_{\nu_j'}(\tau_j')\right]_{B,F}\dots\rangle_0$$

$\sum_{P_{ij}}$ is the sum over permutations that do NOT involve c_{ν_i} and $c^\dagger_{\nu_j}$.

$$\partial^{\theta}_{\tau_i} \mathcal{G}_0^{(n)} = (-1)^n \sum_j (\pm 1)^{i+j} \delta(\tau_i - \tau_j') \delta_{\nu_i \nu_j'} (-1)^{1-n} \mathcal{G}_0^{(n-1)} (\{\nu_n \tau_n\} \setminus \nu_i \tau_i ; \{\nu_n' \tau_n'\} \setminus \nu_j' \tau_j')$$

$$\partial^{n-\theta}_{\tau_i} \mathcal{G}_0^{(n)} = - \sum_\mu h_{\nu_i \mu} \mathcal{G}_0^{(n)} (\nu_1 \tau_1, \dots, \nu_{i-1} \tau_{i-1}, \mu \tau_i, \nu_{i+1} \tau_{i+1}, \dots, \nu_n \tau_n ; \nu_1' \tau_1' \dots \nu_n' \tau_n')$$

$$- \partial_{\tau_i} \mathcal{G}_0^{(n)} - \sum_\mu h_{\nu_i \mu} \mathcal{G}_0^{(n)} (\{\nu_n \tau_n\} \, \nu_i = \mu, \{\nu_n' \tau_n'\}) = - \partial^{\theta}_{\tau_i} \mathcal{G}_0^{(n)}$$

$$= + \sum_j (\pm 1)^{i+j} \delta(\tau_i - \tau_j') \delta_{\nu_i \nu_j'} \mathcal{G}_0^{(n-1)} (\{\nu_n \tau_n\} \setminus \nu_i \tau_i ; \{\nu_n' \tau_n'\} \setminus \nu_j' \tau_j')$$

$$- \sum_\mu \left(\partial_{\tau_i} \delta_{\nu_i \mu} + h_{\nu_i \mu} \right) \mathcal{G}_0^{(n)} (\{\nu_n \tau_n\} \, \nu_i = \mu, \{\nu_n' \tau_n'\}) =$$

$$= + \sum_j (\pm 1)^{i+j} (-1) \sum_\mu \left(\partial_{\tau_i} \delta_{\nu_i \mu} + h_{\nu_i \mu} \right) \mathcal{G}_0^{(1)} (\mu \tau_i ; \nu_j' \tau_j') \, \mathcal{G}_0^{(n-1)} (\{\} \setminus i ; \{\} \setminus j)$$

The solution of the previous equation is, for a generic ν_i

$$\mathcal{G}_0^{(n)} (\{\nu_n \tau_n\}, \{\nu_n' \tau_n'\}) = \sum_j (\pm 1)^{i+j} \mathcal{G}_0^{(1)} (\nu_i \tau_i ; \nu_j' \tau_j') \, \mathcal{G}_0^{(n-1)} (\{\nu_n \tau_n\} \setminus i ; \{\nu_n' \tau_n'\} \setminus j)$$

which is the definition of determinant (permanent) contained in the formulation of Wick's theorem.

fermions: $(-1)^x = -(-1)^n(-1)^{1-n}(-1)^{2n-i-j} = (-1)^{j+i}$

bosons $(-1)^x \quad -(-1)^n(-1)^{1-n} = 1$

$$\Rightarrow \quad \mathcal{G}_0^{(n)}(\nu_1\tau_1,\dots,\nu_n\tau_n;\nu_1'\tau_1',\dots\nu_n'\tau_n') =$$

$$= \sum_{j=1}^{n} (\pm 1)^{j+i}\, \mathcal{G}_0(\nu_i\tau_i,\nu_j'\tau_j')\, \mathcal{G}_0^{(n-1)}(\underbrace{\nu_1\tau_1,\dots,\nu_n\tau_n}_{\text{without } i};\underbrace{\nu_1'\tau_1',\dots,\nu_n'\tau_n'}_{\text{without } j})$$

but this is the definition of permanent (determinant) of the matrix of single particle Matsubara particles.

QED

Feynman diagrams and external potentials

A) single particle but one easy and one complicate:

$$H = H_0 + V = \sum_\sigma \int d\vec{r}\; \psi_\sigma^+(r) H_0(r) \psi_\sigma(r) + \sum_\sigma \int dr\; \psi_\sigma^+(r) V_\sigma(r) \psi_\sigma(r)$$

notation $\quad (r_1,\sigma_1,\tau_1) = (1) \qquad \int d1 = \sum_{\sigma_1}\int dr_1 \int_0^\beta d\tau_1$

$$[-\partial_{\tau_b} - H_0(b)]\,\mathcal{G}^0(b,a) = \delta(b-a) \iff [-\partial_{\tau_b} - H(b) + V(b)]\,\mathcal{G}^0(b,a) = \delta(b-a)$$

$$[-\partial_{\tau_b} - H(b)]\,\mathcal{G}(b,a) = \delta(b-a) \iff \mathcal{G}(b,a) = [-\partial_{\tau_b} - H(b)]^{-1}\delta(b-a)$$

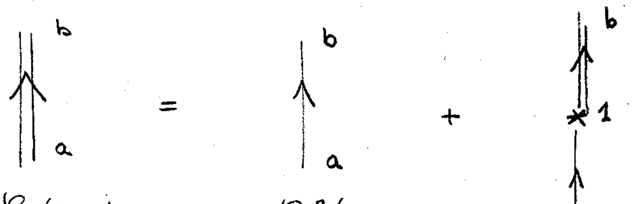
$$\Rightarrow \quad \mathcal{G}(b,a) = [-\partial_{\tau_b} - H(b)]^{-1}[-\partial_{\tau_b} - H(b) + V(b)]\,\mathcal{G}^0(b,a)$$

$$= \mathcal{G}^0(b,a) + \int d1\; \mathcal{G}(b,1)\, V(1)\, \mathcal{G}^0(1,a)$$

$\sim$ Dyson equation

SEE ALSO 179-b 179-c

Graphically:



$\mathcal{G}(b,a) \qquad \mathcal{G}^0(b,a) \qquad \int d1\; \mathcal{G}(b,1)\, V(1)\, \mathcal{G}^0(1,a)$

Dyson equation for external potentials

$$H = H_0 + V = \sum_\sigma \int d\vec{r}\, \psi_\sigma^+(\vec{r})\, H_0(\vec{r})\, \psi_\sigma(\vec{r}) + \sum_\sigma \int d\vec{r}\, \psi_\sigma^+(\vec{r})\, V_\sigma(\vec{r})\, \psi_\sigma(\vec{r})$$

N.B.

$V_\sigma(\vec{r}) = \langle \vec{r}\sigma | V(\hat{\vec{r}}\hat{\sigma}) | \vec{r}\sigma \rangle$ the general steps to the field operator formulation are thus:

$$\hat{V}^{I} = V(\hat{r}\hat{\sigma})$$

$$\hat{V}^{II} = \sum_{\sigma\sigma'} \int d\vec{r}\, d\vec{r}\,' \langle \vec{r}\sigma | V(\hat{r}\hat{\sigma}) | \vec{r}\,'\sigma' \rangle\, \psi_\sigma^+(\vec{r})\, \psi_{\sigma'}(\vec{r}\,')$$

$$= \sum_{\sigma\sigma'} \int d\vec{r}\, d\vec{r}\,' \underbrace{\langle \vec{r}\sigma | \vec{r}\sigma' \rangle}_{\delta(\vec{r}-\vec{r}\,')\delta_{\sigma\sigma'}} V_{\sigma'}(r')\, \psi_\sigma^+(\vec{r})\, \psi_{\sigma'}(\vec{r}\,') = \sum_\sigma \int dr\, \psi_\sigma^+(\vec{r})\, V_\sigma(\vec{r})\, \psi_\sigma(\vec{r})$$

$$\mathcal{G}_0(b,a) \equiv -\langle T_\tau [\, \psi_{0,\sigma_b}(\vec{r}_b\tau_b)\, \psi^+_{0,\sigma_a}(\vec{r}_a\tau_a)] \rangle_0$$

both the evolution and the average are only with respect to H_0

$$\mathcal{G}(b,a) \equiv -\langle T_\tau [\, \psi_{\sigma_b}(\vec{r}_b\tau_b)\, \psi^+_{\sigma_a}(\vec{r}_a\tau_a)] \rangle$$

both evolution and average with respect to H

$$\partial_{\tau_b} \mathcal{G}_0(b,a) = -\partial_{\tau_b} \Big[\theta(\tau_b-\tau_a) \langle \psi_{0,\sigma_b}(\vec{r}_b\tau_b)\, \psi^+_{0,\sigma_a}(\vec{r}_a\tau_a) \rangle_0 + (\pm 1)\, \theta(\tau_a-\tau_b) \langle \psi^+_{0,\sigma_a}(\vec{r}_a\tau_a)\, \psi_{0,\sigma_b}(\vec{r}_b\tau_b) \rangle_0 \Big] =$$

$$= -\delta(\tau_b-\tau_a)\, \delta(\vec{r}_b-\vec{r}_a)\, \delta_{\sigma_b\sigma_a} - H_0(\vec{r}_b)\, \mathcal{G}_0(b,a)$$

$$[-\partial_{\tau_b} - H_0(\vec{r}_b)]\, \mathcal{G}_0(b,a) = \delta(b-a) \equiv \delta(\vec{r}_b-\vec{r}_a)\, \delta(\tau_b-\tau_a)\, \delta_{\sigma_b\sigma_a}$$

In a similar way

$$[-\partial_{\tau_b} - H_0(\vec{r}_b) - V_{\sigma_b}(\vec{r}_b)]\, \mathcal{G}(b,a) = \delta(b-a)$$

$$\left[-\partial_{\tau_b} - H_0(\vec{r}_b)\right] \mathcal{G}_0(b,a) = \left[-\partial_{\tau_b} - H_0(\vec{r}_b) - V_{\sigma_b}(\vec{r}_b)\right] \mathcal{G}(b,a)$$

$$\left[-\partial_{\tau_b} - H_0(\vec{r}_b)\right] \mathcal{G}(b,a) = \left[-\partial_{\tau_b} - H_0(\vec{r}_b)\right] \mathcal{G}^0(b,a) + V_{\sigma_b}(\vec{r}_b)\, \mathcal{G}(b,a)$$

$$V_{\sigma_b}(\vec{r}_b)\, \mathcal{G}^0(b,a) = \int dr_1 \sum_{\sigma_1} \int_0^\beta d\tau_1 \;\; \delta(\vec{r}_b - \vec{r}_1)\, \delta(\tau_b - \tau_1)\, \delta_{\sigma_b \sigma_1} V_{\sigma_1}(\vec{r}_1)\, \mathcal{G}(1,a)$$

$$= \int d1 \;\; \delta(b-1)\, V(1)\, \mathcal{G}(1,a)$$

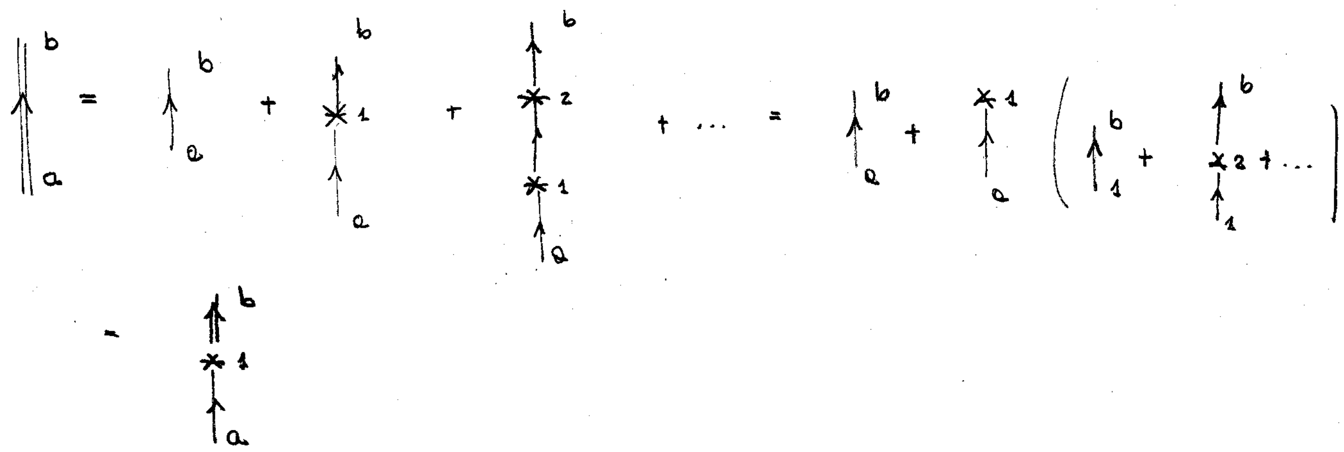
$$= \int d1 \; \left[-\partial_{\tau_b} - H_0(\vec{r}_b)\right] \mathcal{G}^0(b,1)\, V(1)\, \mathcal{G}(1,a)$$

$$= \left[-\partial_{\tau_b} - H_0(\vec{r}_b)\right] \int d1 \;\; \mathcal{G}^0(b,1)\, V(1)\, \mathcal{G}(1,a)$$

$$-\cancel{\left[\partial_{\tau_b} + H_0(b)\right]}\, \mathcal{G}(b,a) = -\cancel{\left[\partial_{\tau_b} + H_0(b)\right]} \left[\mathcal{G}^0(b,a) + \int d1 \; \mathcal{G}^0(b,1)\, V(1)\, \mathcal{G}(1,a)\right]$$

$$\boxed{\mathcal{G}(b,a) = \mathcal{G}^0(b,a) + \int d1 \; \mathcal{G}^0(b,1)\, V(1)\, \mathcal{G}(1,a)}$$

All the derivation has a clear diagrammatic picture:



Elastic scattering and Matsubara frequencies: In the scattering against a static external potential no energy is transferred from the electrons to the "impurities". ⇒ the Matsubara frequency is the same for the all diagram.

recall
$$\mathcal{G}(r\tau, r'\tau') = \frac{1}{\beta}\sum_n \mathcal{G}(r, r'; ik_n)\, e^{-ik_n(\tau-\tau')}$$
$$\mathcal{G}(r, r'; ik_n) = \int_0^\beta d\tau\, \mathcal{G}(r\tau, r'\tau')\, e^{ik_n(\tau-\tau')}$$

Dyson equation:
$$\mathcal{G}(r_b, r_a; ik_n) = \int dr_1\, \mathcal{G}(r_b, r_1; ik_n)\, V(1)\, \mathcal{G}^0(r_1, r_a; ik_n) + \mathcal{G}^0(r_b, r_a; ik_n)$$

In ν space
$$\mathcal{G}(\nu_b\nu_a; ik_n) = \delta_{\nu_b\nu_a}\, \mathcal{G}^0(\nu_a\nu_a; ik_n) + \sum_{\nu_c} \mathcal{G}(\nu_b\nu_c; ik_n)\, V_{\nu_c\nu_a}\, \mathcal{G}^0(\nu_a\nu_a; ik_n)$$

where $V_{\nu_c\nu_a} \equiv \int dr\, \langle \nu_c | r\rangle\, V(r)\, \langle r | \nu_a\rangle$

<u>Random impurities and self-averaging</u>

$$V(r) = \sum_{j=1}^{N_{imp}} u(r - R_j) \qquad R_j \text{ randomly distributed}$$

Formidable task. But 2 dimensionless parameters serve as guide:

$$\frac{n_{imp}}{n_{el}} \ll 1 \qquad\qquad \tilde{u}\,\frac{ma^2}{\hbar^2} \ll \min(1, k_F a)$$

[$\tilde{u}$ tipical value of u in the vicinity of R_j]

The second relation can be read as

$$\underbrace{\tilde{u} \ll \frac{\hbar^2}{ma^2}}_{\text{size quant level spacing}} \qquad\qquad \underbrace{\tilde{u} \ll \frac{\hbar^2 k_F}{ma} = \frac{p_F}{m}\frac{\hbar}{a}}_{\text{level spacing at fermi energy.}}$$

The Dyson equation reads:

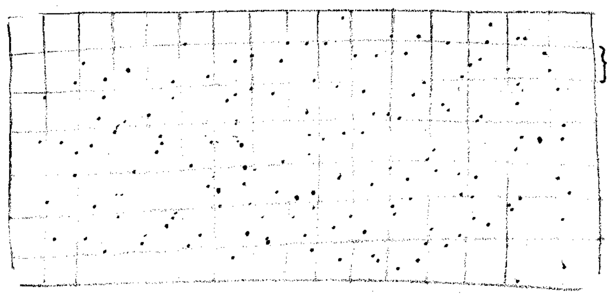
$$\mathcal{G}(r_b, r_a; ik_n) = \mathcal{G}^0(r_b - r_a; ik_n) + \sum_{j=1}^{N_{imp}} \int dr_1\, \mathcal{G}(r_b, r_1; ik_n)\, u(r_1 - P_j)\, \mathcal{G}^0(r_1 - r_a; ik_n)$$

=> the n^{th} order contribution reads:

$$\mathcal{G}^{(n)}(r_b, r_a) = \sum_{j_1=1}^{N_{imp}} \cdots \sum_{j_n=1}^{N_{imp}} \int dr_1 \cdots \int dr_n$$

$$\mathcal{G}^0(r_b - r_n)\, u(r_n - P_{j_n}) \cdots u(r_2 - P_{j_2})\, \mathcal{G}^0(r_2 - r_1)\, u(r_1 - P_{j_1})\, \mathcal{G}^0(r_1 - r_a)$$

Since we do not know the positions of the impurities, we average over them. Physical meaning?



The transport through the system is an incoherent composition of the transport through the phase coherent cells. But the position of the impurities per cell is random, by varying cells. Since we are interested into the average picture given by a coarse grained σ => we average the position of the impurities. At this point also the position of the cells is not important => we average over the entire volume. The averaging restores translation invariance => a momentum description is usefull:

$$\mathcal{G}^{(n)}(r_b, r_a) = \frac{1}{V^2} \sum_{k_a k_b} e^{ik_b r_b} e^{-ik_a r_a}\, \mathcal{G}^{(n)}_{k_b k_a}$$

$$\mathcal{G}^0_k(ik_n) = \frac{1}{ik_n - \xi_k} \qquad \mathcal{G}^0(r - r'; ik_n) = \frac{1}{V} \sum_k \mathcal{G}^0_k(ik_n)\, e^{ik(r-r')}$$

$$u(r - P_j) = \frac{1}{V} \sum_q u_q\, e^{iq(r - P_j)} = \frac{1}{V} \sum_q e^{-iq P_j}\, u_q\, e^{iq \cdot r}$$

[remember that for all of these transformations $\frac{1}{V}$ comes from the renormalization given by 2 plane waves...]

$$\mathcal{G}^{(n)}(r_b, r_a) = \sum_{j_1 \cdots j_n}^{N_{imp}} \frac{1}{V^n} \sum_{q_1 \cdots q_n} \frac{1}{V^2} \sum_{k_a k_b} \frac{1}{V^{n-1}} \sum_{k_1 \cdots k_{n-1}} \int dr_1 \cdots \int dr_n$$

$$\mathcal{G}^0_{k_b} u_{q_n} \mathcal{G}^0_{k_{n-1}} u_{q_{n-1}} \cdots u_{q_2} \mathcal{G}^0_{k_1} u_{q_1} \mathcal{G}^0_{k_a} e^{-i(q_n \cdot P_{j_n} + \cdots + q_2 \cdot P_{j_2} + q_1 \cdot P_{j_1})}$$
$$e^{ik_b(r_b - r_n)} e^{iq_n \cdot r_n} e^{ik_{n-1}(r_n - r_{n-1})} \cdots e^{iq_2 r_2} e^{ik_1(r_2 - r_1)} e^{iq_1 r_1} e^{ik_a(r_1 - r_a)}$$

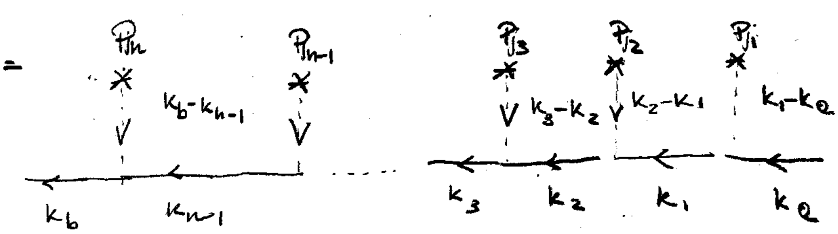
- n spacial integrations can be performed and give n δ-functions
- n q-sum can be performed using the n δ-functions above

$$\mathcal{G}^{(n)}(r_b, r_a) = \frac{1}{V^2} \sum_{k_a k_b} e^{ik_b r_b} e^{-ik_a r_a} \sum_{j_1 \cdots j_n}^{N_{imp}} \sum_{k_1 \cdots k_{n-1}} \frac{1}{V^{n-1}} \cdot$$
$$\mathcal{G}^0_{k_b} u_{k_b - k_{n-1}} \mathcal{G}^0_{k_{n-1}} \cdots u_{k_2 - k_1} \mathcal{G}^0_{k_1} u_{k_1 - k_a} \mathcal{G}^0_{k_a} e^{-i[(k_b - k_{n-1}) P_{j_n} + \cdots + (k_1 - k_a) P_{j_1}]}$$

Eventually we go into k space

$$\mathcal{G}^{(n)}_{k_b k_a} = \sum_{j_1 \cdots j_n}^{N_{imp}} \sum_{k_1 \cdots k_{n-1}} \frac{1}{V^{n-1}} e^{-i[(k_n - k_{n-1}) \cdot P_{j_n} + \cdots + (k_1 - k_a) \cdot P_{j_1}]} \cdot$$
$$\cdot\, \mathcal{G}^0_{k_b} u_{k_b - k_{n-1}} \mathcal{G}^0_{k_{n-1}} \cdots u_{k_2 - k_1} \mathcal{G}^0_{k_1} u_{k_1 - k_a} \mathcal{G}^0_{k_a}$$

$\mathcal{G}^{(n)}_{k_b k_a} =$



$$\langle \cdot \rangle_{imp} = \frac{1}{V^{N_{imp}}} \prod_{j=1}^{N_{imp}} \int dP_j$$

$P_{j_1} \cdots P_{j_n}$ need not to be n different scatterers. Notice that P's are entering only in the exponential factor => we can concentrate on it. The exponential should be organized to separate out independent scatterers:

$$\left\langle \sum_{j_1 \cdots j_n}^{N_{imp}} e^{i \sum_{l=1}^{n} q_l \cdot P_{j_l}} \right\rangle_{imp} = \sum_{p=1}^{n} \left[\sum_{Q_1 \cup Q_2 \cup \ldots \cup Q_p = Q} \prod_{h=1}^{p} N_{imp}\, \delta_{0, \sum_{i,\, q_{h_i} \in Q_h} q_{h_i}} \right]$$

p is # of independent impurities

Q_i is the group of momenta transfered to the same impurity

The fact that the translation invarience is recovered implies that $\langle \mathcal{G}^{(n)} \rangle_{imp}$ depends on 1 momentum only:

$$\langle \mathcal{G}^{(n)}_k \rangle_{imp} = \frac{1}{V^{n-1}} \sum_{k_1 \cdots k_{n-1}} \sum_{p=1}^{n} \sum_{Q_1 \cup Q_2 \cup Q_3 \ldots \cup Q_p = Q} \prod_{h=1}^{p} \left(N_{imp}\, \delta_{0, \sum_i (k_{h_i} - k_{h_{i-1}})} \right)$$

$$\times\ \mathcal{G}^0_k u_{k-k_1} \mathcal{G}^0_{k_1} u_{k_1 - k_2} \mathcal{G}^0_{k_2} \cdots u_{k_{n-1}-k} \mathcal{G}^0_k$$

p δ-functions saturate p sums. $\dfrac{N_{imp}^{\,p}}{V^p} = n_{imp}$

<u>Feynman rules</u> for $\langle \mathcal{G}^{(n)} \rangle_{imp}$

$\boxed{1}$ $\dashleftarrow q$ $= u_q$ (scattering line)

$\boxed{2}$ $*$ $= n_{imp}\, \delta_{0\, \Sigma q}$ (star)

$\boxed{3}$ $\leftarrow k$ $= \mathcal{G}^0_k$ (fermion line)

$\boxed{4}$ Draw p impurity stars. Let n_1 scattering lines go out from star 1, n_2 from star 2 etc. $n_1 + n_2 + \ldots + n_p = n$

$\boxed{5}$ Draw all topologically different diagrams containing an unbroken chain of $n+1$ fermion lines connecting once to each of the n scattering line end-points.

$\boxed{6}$ Let the first and the last fermion line be $\mathcal{G}^0_k$.

$\boxed{7}$ Maintain momentum conservation at each vertex