From Wigner distribution to Boltzmann distribution

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In quantum mechanics a state cannot be characterized by both its position and momentum. => the notion of probability distribution in phase space its problematic:

$$[\hat{q}, \hat{p}] = i\hbar \quad \Rightarrow \quad f(q,p,t) = \;?$$

Nevertheless one can think to introduce the notion of quasi-distribution f_Q compared to f_{CL} in the following sense

A observable	$\hat{A}$ observable
$\langle A \rangle_{CL} = \int dq \int dp \; A(q,p) \, f_{CL}(q,p)$	$\langle \hat{A} \rangle_Q = \mathrm{Tr}\,(\hat{A}\hat{\rho})$

N.B. $\hat{\rho}$ is the density operator for a system: $\hat{\rho} = \sum_n p_n |\psi_n\rangle\langle\psi_n|$ where $\{|\psi_n\rangle\}$ is a given orthonormal basis and $\sum_n p_n = 1$.

The question is: is the

1 – a mapping $A(q,p) \longrightarrow \hat{A}(\hat{q},\hat{p})$

2 – a quasi-distribution function $f_Q(q,p) \longleftarrow \hat{\rho}$

Such that

$$\mathrm{Tr}\,(\hat{\rho}\hat{A}) = \int dq \int dp \; f_Q(q,p) \, A(q,p)$$

Different solutions have been proposed. We will concentrate on

1 – Weyl 1927

2 – Wigner 1932

Tentative solutions (naive)

$$P_{pos}(q) = \mathrm{Tr}\left(\hat\rho\, \delta(q-\hat q)\right)$$
$$P_{mom}(p) = \mathrm{Tr}\left(\hat\rho\, \delta(p-\hat p)\right)$$

where $\delta(q-\hat q)|q'\rangle \equiv |q\rangle\langle q|q'\rangle = \delta(q-q')|q'\rangle$

$$\rho(q',q'') = \langle q'|\hat\rho|q''\rangle = \sum_n p_n \psi_n(q')\psi_n^*(q'')$$

$$\Rightarrow \quad P_{pos}(q) = \int dq' \langle q'|\hat\rho\, \delta(q-\hat q)|q'\rangle = \sum_n p_n |\psi_n(q)|^2$$

$$P_{mom}(p) = \int dp' \langle p'|\hat\rho\, \delta(p-\hat p)|p'\rangle = \langle p|\hat\rho|p\rangle$$

$$= \int dx\, dx' \langle p|x\rangle \rho(x,x')\langle x'|p\rangle =$$

$$= \int \frac{dx\, dx'}{2\pi\hbar} e^{ip(x'-x)/\hbar} \rho(x,x')$$

$$P_Q^{(1)}(q,p) = \mathrm{Tr}\left(\hat\rho\, \delta(q-\hat q)\, \delta(p-\hat p)\right) \qquad ?$$
$$P_Q^{(2)}(q,p) = \mathrm{Tr}\left(\hat\rho\, \delta(p-\hat p)\, \delta(q-\hat q)\right)$$

We could choose one of them, but which? Above all

THEY ARE NOT REAL (Exercise)

proof: $\int dx' \langle x'|\hat\rho\, \delta(q-\hat q)\delta(p-\hat p)|x'\rangle = \int \frac{dx'}{2\pi\hbar} \rho(x',q)\, e^{ip(q-x')/\hbar}$

$\psi(x): \mathbb{R} \to \mathbb{R} \quad \Rightarrow \mathrm{Im}\left(P_Q^{(1)}(q,p)\right) = \int \frac{dx'}{2\pi\hbar} \psi(x')\psi(q) \sin\left[p(q-x')\right]$

$\psi(x) = \sqrt{\frac{\alpha}{\pi}} \sin(\alpha x) \quad [0, 2\pi/\alpha]$

$p = \alpha \qquad \mathrm{Im}\left(P_Q(q,\alpha)\right) = \frac{\alpha}{\pi}\int_0^{2\pi/\alpha} \frac{dx'}{2\pi\hbar} \sin(\alpha x')\sin(\alpha q)\sin(\alpha(q-x')) =$

$$= \frac{\alpha}{\pi}\int_0^{2\pi/\alpha} \frac{dx'}{2\pi\hbar}\ \cancel{\sin(\alpha x')\cos(\alpha x')\sin^2(\alpha q)} - \sin^2(\alpha x')\sin(\alpha q)\cos(\alpha q)$$

$$= -\frac{1}{4\pi\hbar}\sin(2\alpha q) \neq 0 \qquad q \neq$$

Wigner Distribution

- properties

$$f_W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dy \, \langle q - \tfrac{y}{2} | \hat{\rho} | q + \tfrac{y}{2} \rangle \, e^{\frac{ipy}{\hbar}}$$ ⟵ definition [1932]

It is the only one that

i) Real → Exercise.

ii)
$$\int dp \; f_W(q,p) = \langle q | \hat{\rho} | q \rangle$$
$$\int dq \; f_W(q,p) = \langle p | \hat{\rho} | p \rangle$$
$$\int dq \, dp \; f_W(q,p) = \mathrm{Tr}(\hat{\rho}) = 1$$

iii) $f_W(q,p)$ is Galilei invariant $\begin{cases} \psi(q) \to \psi(q+a) \Rightarrow f_W(q,p) \to f_W(q+a,p) \\ \psi(q) \to e^{ip'q/\hbar}\psi(q) \Rightarrow f_W(q,p) \to f_W(q,p-p') \end{cases}$

iv) $f_W(q,p)$ is invariant under space and time reflections

$$\psi(q) \to \psi(-q) \quad \Rightarrow \quad f_W(q,p) \to f_W(-q,-p)$$
$$\psi(q) \to \psi^*(q) \quad \Rightarrow \quad f_W(q,p) \to f_W(q,-p)$$

v) Classical motion in the force free case:

$$\frac{\partial f_W}{\partial t} = -\frac{p}{m}\frac{\partial f_W}{\partial q}$$

vi) If $f_W^{\psi}(q,p)$ and $f_W^{\varphi}(q,p)$ correspond to the states $\psi(q)$ and $\varphi(q)$

$$\Rightarrow \quad \left| \int dq \, \psi^*(q)\varphi(q) \right|^2 = (2\pi\hbar) \int dq \int dp \; f_W^{\psi}(q,p) \, f_W^{\varphi}(q,p)$$

+ f_W cannot be too peaked $\quad \psi = \varphi$

− f_W can also be NEGATIVE $\quad \psi \perp \varphi$

(vii) If $A(q,p) = \int dz \; \langle q - \frac{z}{2} | \hat{A} | q + \frac{z}{2} \rangle \, e^{i pz/\hbar}$

$\Rightarrow \quad \mathrm{Tr}(\hat{\rho}\hat{A}) = \int dq \int dp \; A(q,p) \, f_W(q,p)$

notice that $(2\pi\hbar)^{-1}$ is missing with respect to the definition of $\hat{\rho} \leftrightarrow f_W$.

viii) There is a perfect symmetry between q and p:

$$\phi(p) = \frac{1}{2\pi\hbar} \int dq \; \psi(q) \, e^{-i qp/\hbar}$$

$$\Rightarrow \quad f_W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dp' \; \langle p + \frac{p'}{2} | \hat{\rho} | p - \frac{p'}{2} \rangle \, e^{+i \frac{qp'}{\hbar}}$$

PROOFS: (remember $2\pi\delta(x) = \int_{-\infty}^{+\infty} e^{ikx} dk$ and $\langle x | p \rangle = \frac{1}{2\pi\hbar} e^{i\frac{xp}{\hbar}}$)

i) $f_W - f_W^* \simeq \int_{-\infty}^{+\infty} dy \left[\langle q - \frac{y}{2} | \hat{\rho} | q + \frac{y}{2} \rangle \, e^{i\frac{py}{\hbar}} - \langle q + \frac{y}{2} | \hat{\rho} | q - \frac{y}{2} \rangle \, e^{-i\frac{py}{\hbar}} \right]$

$y \to -y$ in the second element $\Rightarrow$ exact 0, $\forall q, p$.

ii) • $\int dp \; f_W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dy \; \langle q - \frac{y}{2} | \hat{\rho} | q + \frac{y}{2} \rangle \, e^{\frac{ipy}{\hbar}} = \qquad k = \frac{p}{\hbar}$

$$= \frac{1}{2\pi\cancel{\hbar}} \int_{-\infty}^{+\infty} dk \; \cancel{\hbar} \, e^{iky} \; \langle q - \frac{y}{2} | \hat{\rho} | q + \frac{y}{2} \rangle = \langle q | \hat{\rho} | q \rangle$$

• $\int dq \; f_W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dy \; \langle q - \frac{y}{2} | \hat{\rho} | q + \frac{y}{2} \rangle \, e^{i\frac{py}{\hbar}} =$

$$= \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dp_1 \int_{-\infty}^{+\infty} dp_2 \; e^{i(q - \frac{y}{2})p_1/\hbar \, - \, i(q + \frac{y}{2})p_2/\hbar \, + \, i\frac{py}{\hbar}} \langle p_1 | \hat{\rho} | p_2 \rangle$$

$= \langle p | \hat{\rho} | p \rangle$ it is all a game of δ functions. Make the integrals in the order q, p_1, y, p_2.

• $\int dq\,dp \; f_W(q,p) = \mathrm{Tr}(\hat{\rho}) = 1$ it follows from the previous one

iii) $$f_W(q,p) = \sum_n p_n \int \frac{dy}{2\pi\hbar} \psi_n\left(q-\frac{y}{2}\right)\psi_n^*\left(q+\frac{y}{2}\right) e^{i\frac{py}{\hbar}}$$

- $\bar{\psi}_n(q) = \psi_n(q+a)$

$$\bar{f}_W(q,p) = \sum_n p_n \int \frac{dy}{2\pi\hbar} \bar{\psi}_n\left(q-\frac{y}{2}\right)\bar{\psi}_n^*\left(q+\frac{y}{2}\right) e^{i\frac{py}{\hbar}}$$

$$= \sum_n p_n \int \frac{dy}{2\pi\hbar} \psi_n\left(q-\frac{y}{2}+a\right)\psi_n^*\left(q+\frac{y}{2}+a\right) e^{\frac{ipy}{\hbar}} = f_W(q+a,p)$$

- $\bar{\psi}_n(q) = e^{ip'q/\hbar}\psi_n(q)$

$$\bar{f}_W(q,p) = \sum_n p_n \int \frac{dy}{2\pi\hbar} \bar{\psi}_n\left(q-\frac{y}{2}\right)\bar{\psi}_n^*\left(q+\frac{y}{2}\right) e^{\frac{ipy}{\hbar}} =$$

$$= \sum_n p_n \int \frac{dy}{2\pi\hbar} \psi_n\left(q-\frac{y}{2}\right)\psi_n^*\left(q+\frac{y}{2}\right) e^{i\frac{p'}{\hbar}\left(\cancel{q}-\frac{y}{2}-\cancel{q}-\frac{y}{2}\right)+i\frac{py}{\hbar}} =$$

$$= f_W(q,p-p')$$

iv) $\bar{\psi}_n(q) = \psi_n(-q)$

$$\bar{f}_W(q,p) = \sum_n p_n \int \frac{dy}{2\pi\hbar} \bar{\psi}_n\left(q-\frac{y}{2}\right)\bar{\psi}_n^*\left(q+\frac{y}{2}\right) e^{\frac{ipy}{\hbar}} =$$

$$= \sum_n p_n \int \frac{dy}{2\pi\hbar} \psi_n\left(-q+\frac{y}{2}\right)\psi_n^*\left(-q-\frac{y}{2}\right) e^{\frac{ipy}{\hbar}} =$$

$$= \sum_n p_n \int \frac{dy}{2\pi\hbar} \psi_n\left(-q-\frac{y}{2}\right)\psi_n^*\left(-q+\frac{y}{2}\right) e^{-\frac{ipy}{\hbar}} = f_W(-q,-p)$$

$\bar{\psi}_k(q) = \psi_k^*(q)$

$$\bar{f}_W(q,p) = \sum_n p_n \int \frac{dy}{2\pi\hbar} \psi_n^*\left(q-\frac{y}{2}\right)\psi_n\left(q+\frac{y}{2}\right) e^{\frac{ipy}{\hbar}} =$$

$$= \sum_n p_n \int \frac{dy}{2\pi\hbar} \psi_n\left(q-\frac{y}{2}\right)\psi_n^*\left(q+\frac{y}{2}\right) e^{-\frac{ipy}{\hbar}} = f_W(q,-p)$$

v) $$\frac{\partial f_W}{\partial t} = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dy \left\langle q-\frac{y}{2}\right|\dot{\hat{\rho}}\left|q+\frac{y}{2}\right\rangle e^{i\frac{py}{2}}$$

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[H,\hat{\rho}] = -\frac{i}{\hbar}\left[\frac{\hat{p}^2}{2m},\hat{\rho}\right]$$

$$= \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty} dy\, dp_1\, dp_2\; e^{ip_1(q-\frac{y}{2})/\hbar - ip_2(q+\frac{y}{2})/\hbar + i\frac{py}{2}} \left[-\frac{i}{\hbar}\frac{p_1^2}{2m}+\frac{i}{\hbar}\frac{p_2^2}{2m}\right]$$

$$\cdot \langle p_1|\hat{\rho}|p_2\rangle$$

first we perform the integral in y $\qquad 2\pi\hbar\,\delta\left(p-\frac{p_1}{2}-\frac{p_2}{2}\right)$

$$p_2 = 2p - p_1$$

from the δ

$$\frac{\partial f_w}{\partial t} = \frac{\boxed{2}}{2\pi\hbar}\int dp_1 \; e^{ip_1 q/\hbar - i(2p-p_1)q/\hbar}\left(-\frac{i}{\hbar}\right)\left[\frac{\cancel{p_1^2}}{2m} + \frac{4p^2}{2m} - \frac{\cancel{p_1^2}}{2m} + \frac{4pp_1}{2m}\right]$$

$$\langle p_1|\hat{\rho}|2p-p_1\rangle$$

$$= \frac{2}{2\pi\hbar}\int dp_1 \; e^{i2q(p_1-p)/\hbar}\left(\frac{-2i}{\hbar}\right)\frac{p}{m}(p_1-p) \quad \langle p_1|\hat{\rho}|2p-p_1\rangle$$

$$= -\frac{p}{m}\frac{\partial}{\partial q}\;\frac{2}{2\pi\hbar}\int dp_1 \; e^{i2q(p_1-p)/\hbar}\langle p_1|\hat{\rho}|2p-p_1\rangle$$

$$= -\frac{p}{m}\frac{\partial}{\partial q}\frac{2}{(2\pi\hbar)^2}\int dp_1\,dx\,dy \; e^{i[p_1(2q-x-y)+p(-2q+2y)]\frac{1}{\hbar}}\langle x|\hat{\rho}|y\rangle$$

$$= -\frac{p}{m}\frac{\partial}{\partial q}\frac{4}{2\pi\hbar}\int dx\,dy \;\; \delta(2q-x-y)\; e^{ip(2y-2q)/\hbar}\langle x|\hat{\rho}|y\rangle$$

$$= -\frac{p}{m}\frac{\partial}{\partial q}\frac{4}{2\pi\hbar}\int dy \; e^{i2p(y-q)/\hbar}\langle 2q-y|\hat{\rho}|y\rangle \qquad \bar{y} = 2(y-q), \quad y = \frac{\bar{y}}{2}+q$$

$$= -\frac{p}{m}\frac{\partial}{\partial q}\frac{\cancel{4}}{2\pi\hbar}\cdot\frac{1}{\cancel{2}}\int d\bar{y} \;\; \langle q-\tfrac{\bar{y}}{2}|\hat{\rho}|q+\tfrac{\bar{y}}{2}\rangle \; e^{ip\bar{y}/\hbar}$$

$$= -\frac{p}{m}\frac{\partial}{\partial q} f_w(q,p).$$

vi)

$$2\pi\hbar\int dq\int dp \; f_w^{\psi}(q,p)\,f_w^{\varphi}(q,p) =$$

$$= \frac{1}{2\pi\hbar}\int dq\,dp\int dy_1\int dy_2 \;\; \psi(q-\tfrac{y_1}{2})\,\psi^*(q+\tfrac{y_1}{2})\;\varphi(q-\tfrac{y_2}{2})\,\varphi^*(q+\tfrac{y_2}{2})\; e^{ip(y_1+y_2)/\hbar}$$

$$= \int dq\int dy \;\; \psi(q-\tfrac{y}{2})\,\psi^*(q+\tfrac{y}{2})\,\varphi(q+\tfrac{y}{2})\,\varphi^*(q-\tfrac{y}{2}) =$$

$$= \int dq\,dy \;\; F(q,y) = \int dy\int dq \;\; F(q+\tfrac{y}{2},y)$$

$$= \int dy\int dq \;\; \psi(q)\,\varphi^*(q)\;\;\psi^*(q+y)\,\varphi(q+y)$$

$$= \int dq\;\psi(q)\varphi^*(q)\int dy\;\psi^*(y)\varphi(y) = \left|\int dq\;\psi(q)\varphi^*(q)\right|^2$$

$$\psi = \varphi \qquad \int dq\, dp\; f_w^2(q,p) = (2\pi\hbar)^{-1} \quad \text{and}$$

$$\psi \perp \varphi \qquad \int dq\, dp\; f_w^{\psi}(q,p)\, f_w^{\varphi}(q,p) = 0$$

vii)

$$A(q,p) = \int dz \; \langle q - \tfrac{z}{2} | \hat{A} | q + \tfrac{z}{2} \rangle \, e^{ipz/\hbar}$$

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \int dx \, \langle x | \hat{\rho}\hat{A} | x \rangle = \int dx\, dy \; \rho(x,y)\, A(y,x)$$

$$= \int d\bar{x}\, d\Delta x \;\; \rho\left(\bar{x} + \tfrac{\Delta x}{2}, \bar{x} - \tfrac{\Delta x}{2}\right) A\left(\bar{x} - \tfrac{\Delta x}{2}, \bar{x} + \tfrac{\Delta x}{2}\right) \qquad \bar{x} = \frac{x+y}{2}, \quad \Delta x = x - y$$

$$\rho\left(\bar{x} + \tfrac{\Delta x}{2}, \bar{x} - \tfrac{\Delta x}{2}\right) = \int dp \; f_w(\bar{x}, p)\, e^{-ip\Delta x/\hbar}$$

$$= \int d\bar{x}\, d\Delta x\, dp \;\; f_w(\bar{x},p)\, e^{-ip\Delta x/\hbar} \, A\left(\bar{x} - \tfrac{\Delta x}{2}, \bar{x} + \tfrac{\Delta x}{2}\right)$$

$$= \frac{1}{2\pi\hbar} \int d\bar{x}\, d\Delta x\, dp\, dp' \;\; f_w(\bar{x},p)\, e^{-ip\frac{\Delta x}{\hbar}} \, A(\bar{x}, p')\, e^{ip'\frac{\Delta x}{\hbar}}$$

$$\int \Delta x, \int p', \quad \bar{x} \to q.$$

$$= \int dq\, dp \;\; f_w(q,p)\, A(q,p)$$

It is a very important relation, "almost" validate the introduction of the Wigner function.

$$A(q,p) \quad \text{of} \quad \hat{A} = \sum_n \left[\alpha_n \hat{p}^n + \beta_n \hat{q}^n \right]$$

$$A(q,p) = \int dz \;\; \langle q - \tfrac{z}{2} | \sum_n \alpha_n \hat{p}^n + \beta_n \hat{q}^n | q + \tfrac{z}{2} \rangle \, e^{ipz/\hbar}$$

$$= \sum_n \Big[\alpha_n \int dz \;\; \langle q - \tfrac{z}{2} | \hat{p}^n | q + \tfrac{z}{2} \rangle \, e^{ipz/\hbar} +$$

$$+ \beta_n \int dz \;\; \langle q - \tfrac{z}{2} | q + \tfrac{z}{2} \rangle \left(q + \tfrac{z}{2}\right)^n e^{ipz/\hbar} \Big]$$

$$A(q,p) = \sum_n \left[\alpha_n \int dz \int dp' \; p'^n \frac{e^{iz(p-p')/\hbar}}{2\pi\hbar} + \beta_n \int dz \; \delta(z) \left(q+\frac{z}{2}\right)^n e^{ipz/\hbar} \right] = \sum_n \alpha_n p^n + \beta_n q^n .$$

The correspondence introduced by Wigner gives an obvious result in the case of sums of $\hat{q}$-polynomials and $\hat{p}$-polynomials.

The difficult point is still somehow open of constructing a quantum operator from a classical one that preserves the property vii).

viii)

$$f_W(q,p) = \frac{1}{2\pi\hbar} \int dy \, \langle q - \tfrac{y}{2} | \hat{\rho} | q + \tfrac{y}{2} \rangle \, e^{ipy/\hbar}$$

$$= \frac{1}{(2\pi\hbar)^2} \int dy \, dp_1 dp_2 \; e^{ip_1(q-\frac{y}{2})\frac{1}{\hbar}} \langle p_1 | \hat{\rho} | p_2 \rangle \, e^{-i\frac{p_2}{\hbar}(q+\frac{y}{2}) + i\frac{py}{\hbar}}$$

$$\int dy \longrightarrow 2\pi\hbar \, \delta\left(-\frac{p_1}{2} - \frac{p_2}{2} + p\right) \qquad p_1 = 2p - p_2$$

$$= \frac{1}{2\pi\hbar} \int dp_2 \; 2 \; e^{i(2p-p_2)\frac{q}{\hbar}} \langle 2p - p_2 | \hat{\rho} | p_2 \rangle \, e^{-\frac{ip_2 q}{\hbar}}$$

$$2p - 2p_2 = \bar{p} \qquad p_2 = -\frac{\bar{p}}{2} + p$$

$$= \frac{1}{2\pi\hbar} \int d\bar{p} \, \langle p + \tfrac{\bar{p}}{2} | \hat{\rho} | p - \tfrac{\bar{p}}{2} \rangle \, e^{i\frac{\bar{p}q}{\hbar}}$$

It is possible to show that properties i-v determine the Wigner function uniquely. Alternatively i-iv and vi do the same job.

The property (vii) leads to a very important question: given $\hat{A}$ and $\hat{B}$ two quantum mechanical operators and $A(q,p)$ and $B(q,p)$ the associated variables in the phase space ...

continue

... $\hat{F} = \hat{A}\hat{B}$ and $F(q,p)$ the associated variable in μ:

$$F(q,p) \stackrel{?}{=} A(q,p) \, , \, B(q,p)$$

In order to solve this problem we have to introduce a second one, that is the relation between a classical variable and the associated quantum mechanical operator.

* Quantum measurement is the projection of a quantum state $|\psi\rangle$ into one of the eigenstates of the operator $\hat{O}$ with a probability given by $|\langle\psi|\psi_\lambda\rangle|^2$ when $\hat{O}|\psi_\lambda\rangle = O_\lambda|\psi_\lambda\rangle$.

* Classically? The state of the system is characterized by q,p and $\mathcal{O}(q,p)$ is a number associated with the configuration q,p. But we postulate ...

Weyl: $\quad A_{cl}(q,p) \longrightarrow \hat{A}$

It is remarkable that with the Weyl association

$$\langle\psi|\hat{A}|\psi\rangle = \int dq \int dp \; f_W(q,p)\, A(q,p)$$

where $f_W(q,p)$ is the Wigner function and $A(q,p)$ is the classical observable. $(\hbar = 1)$

$$A(q,p) = \int d\sigma \int d\tau \;\; \alpha(\sigma,\tau)\, e^{i(\sigma q + \tau p)}$$

$$\rightarrow \hat{A}(\hat{q},\hat{p}) = \int d\sigma \int d\tau \;\; \alpha(\sigma,\tau)\, e^{i(\sigma \hat{q} + \tau \hat{p})}$$

Theorem : $A(q,p)$ classical $\xrightarrow{\text{Weyl}}$ $\hat{A}(\hat{q},\hat{p})$.

$$\hat{\rho} \longrightarrow f_W(q,p)$$

$$\Rightarrow \quad \mathrm{Tr}(\hat{\rho}\hat{A}) = \int dq\, dp\; A(q,p) f_W(q,p) \qquad *$$

proof

Let's assume that (*) is valid for pure states $\Rightarrow$ it is valid for any state

$$\hat{\rho} = \sum_n p_n |\psi_n\rangle\langle\psi_n| \qquad \mathrm{Tr}(\hat{\rho}\hat{A}) = \sum_n \mathrm{Tr}(|\psi_n\rangle\langle\psi_n|\hat{A})$$

$$= \sum_n p_n \int dq\, dp\; A(q,p)\, f_W^{\psi_n}(q,p) = \int dq\, dp\; A(q,p) \sum_n p_n f_W^{\psi_n}(q,p)$$

$$= \int dq\, dp\; A(q,p)\, f_W(q,p).$$

We have to prove it now for pure states

$$\langle\psi|\hat{A}(\hat{q},\hat{p})|\psi\rangle = \langle\psi| \int d\tau \int d\sigma\; \alpha(\sigma,\tau)\, e^{i(\sigma\hat{q}+\tau\hat{p})}|\psi\rangle$$

$$= \int d\tau \int d\sigma\; \alpha(\sigma,\tau)\, \langle\psi|e^{i(\sigma\hat{q}+\tau\hat{p})}|\psi\rangle$$

the relevant step

$$\overset{!}{=} \int d\tau \int d\sigma\; \alpha(\sigma,\tau) \int dq \int dp\; e^{i(\sigma q+\tau p)} f_W^{\psi}(q,p)$$

$$= \int dq\, dp\; A(q,p)\, f_W(q,p)$$

$\boxed{\text{LHS}}$

$$\langle\psi|e^{i(\sigma\hat{q}+\tau\hat{p})}|\psi\rangle = \langle\psi|e^{i\sigma\hat{q}}\, e^{i\tau\hat{p}}\, e^{i\sigma\tau/2}|\psi\rangle =$$

$$= e^{\frac{i\sigma\tau}{2}}\langle\psi|e^{i\sigma\hat{q}}\, e^{i\tau\hat{p}}|\psi\rangle = e^{\frac{i\sigma\tau}{2}}\int dx\; \langle\psi|e^{i\sigma\hat{q}}|x\rangle\langle x|e^{i\tau\hat{p}}|\psi\rangle$$

$$= e^{i\sigma\tau/2}\int dx\; \psi^*(x)\, e^{i\sigma x}\, \psi(x+\tau) = \int dx\; e^{i\sigma\tau/2 + i\sigma x}\, \psi^*(x)\psi(x+\tau)$$

$\boxed{\text{RHS}}$

$$\int dq\, dp\; e^{i(\sigma q+\tau p)} f_W^{\psi}(q,p) = \int dq\, dp\; e^{i(\sigma q+\tau p)} \int \frac{dy}{2\pi}\, \psi\left(q-\frac{y}{2}\right)\psi^*\left(q+\frac{y}{2}\right) e^{ipy}$$

$$= \int dq\; e^{i\sigma q}\, \psi\left(q+\frac{\tau}{2}\right)\psi^*\left(q-\frac{\tau}{2}\right) = \int dx\; e^{i\sigma x + i\frac{\sigma\tau}{2}}\, \psi^*(x)\,\psi(x+\tau) \qquad \blacksquare$$

$$q-\tau/2 = x$$

If $A_{cl}(q,p) \to \hat{A}(\hat{q},\hat{p}) = \int d\sigma\, d\tau\; e^{i\sigma\hat{q}+i\tau\hat{p}}\,\alpha(\sigma,\tau)$

$$A_W(q,p) = \int dz\, \langle q-\tfrac{z}{2}|\hat{A}|q+\tfrac{z}{2}\rangle\, e^{ipz/\hbar} = A_{cl}(q,p)$$

Exercise: Prove that

$$q^n p^m \to \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\hat{q}^{n-r}\hat{p}^m\hat{q}^r$$

$$F = q^n p^m \qquad \hat{F}_{Weyl} = \int d\sigma\, d\tau\; e^{i(\sigma\hat{q}+\tau\hat{p})/\hbar}\,\varphi(\sigma,\tau)$$

where $F(q,p) = \int d\sigma\, d\tau\; e^{i(\sigma q+\tau p)/\hbar}\,\varphi(\sigma,\tau)$

$$\Rightarrow \varphi(\sigma,\tau) = \frac{1}{(2\pi\hbar)^2}\int dq\, dp\; e^{-i(\sigma q+\tau p)/\hbar}\, q^n p^m$$

$$= \frac{1}{(2\pi\hbar)^2}\frac{\hbar^{n+m}}{(-i)^{n+m}}\frac{\partial^n}{\partial\sigma^n}\frac{\partial^m}{\partial\tau^m}\int dq\, dp\; e^{-i(\sigma q+\tau p)} = (i\hbar)^{n+m}\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}\delta(\sigma)\delta(\tau)$$

$$\hat{F}_{Weyl} = \int d\sigma\, d\tau\; (i\hbar)^{n+m}\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}\delta(\sigma)\delta(\tau)\; e^{i(\sigma\hat{q}+\tau\hat{p})/\hbar}$$

$$= \int d\sigma\, d\tau\; (-1)^{n+m}(i\hbar)^{n+m}\,\delta(\sigma)\delta(\tau)\,\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}\, e^{i(\sigma\hat{q}+\tau\hat{p})/\hbar}$$

$$= \frac{1}{(n+m)!}\frac{\partial^{n+m}}{\partial\sigma^n\partial\tau^m}(\sigma\hat{q}+\tau\hat{p})^{n+m}\Big|_{\sigma=0,\tau=0} = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\hat{q}^{n-r}\hat{p}^m\hat{q}^r$$

Eventually the relation $F(a,b) \overset{?}{=} [A(a,b),B(a,b)]$

$$\hat{A} = \int d\sigma\, d\tau\; e^{i(\sigma\hat{q}+\tau\hat{p})}\,\alpha(\sigma,\tau)$$

- $\langle q'|\hat{A}|q''\rangle = \int d\sigma\, d\tau\; \langle q'|e^{i(\sigma\hat{q}+\tau\hat{p})}|q''\rangle\,\alpha(\sigma,\tau) =$

$$= \int d\sigma\, d\tau\; \langle q'|e^{i\sigma\hat{q}}\, e^{i\tau\hat{p}+\frac{\tau\sigma}{2}}|q''\rangle\,\alpha(\sigma,\tau)$$

$$= \int d\sigma\, d\tau\; e^{i\sigma q'+\frac{\tau\sigma}{2}}\langle q'|e^{i\tau\hat{p}}|q''\rangle\,\alpha(\sigma,\tau)$$

$$= \int d\sigma\, d\tau\; e^{i\sigma q'+\frac{\tau\sigma}{2}}\,\delta(q'-\tau-q'')\,\alpha(\sigma,\tau) = \int d\sigma\; e^{i\sigma\frac{q'+q''}{2}}\,\alpha(\sigma,q'-q'')$$

$$\langle q''|\hat{B}|q'\rangle = \int d\sigma \; e^{i\sigma\frac{q'+q''}{2}} \; \beta(\sigma, q'-q'')$$

$$F(q,p) = \int dz \; e^{i/\hbar\, pz} \langle q - \tfrac{z}{2} | \hat{A}\hat{B} | q + \tfrac{z}{2} \rangle$$

$$= \int dz\, dq' \; e^{\frac{i}{\hbar}pz} \langle q - \tfrac{z}{2} | \hat{A} | q' \rangle \langle q' | \hat{B} | q + \tfrac{z}{2} \rangle$$

$$= \int dz\, dq'\, d\sigma\, d\sigma' \; e^{i/\hbar\, pz} \; e^{i/\hbar\, \sigma(q'+q-z/2)/2} \; e^{i/\hbar\, \sigma'(q'+q+z/2)/2}$$

$$\alpha(\sigma, \underbrace{q'-q+\tfrac{z}{2}}_{\tau}) \; \beta(\sigma', \underbrace{q+\tfrac{z}{2}-q'}_{\tau'})$$

$$= \int d\tau\, d\tau'\, d\sigma\, d\sigma' \quad e^{i/\hbar(\sigma q + \tau p)} \, \alpha(\sigma,\tau) \, \underbrace{e^{i/\hbar(\sigma'\tau - \sigma\tau')/2}} \, e^{i/\hbar(\sigma' q + \tau' p)} \beta(\sigma', \tau')$$

$$\frac{i}{\hbar}\sigma'\tau = \frac{\hbar}{i}\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} \quad \Rightarrow \quad e^{i/\hbar(\sigma'\tau - \sigma\tau')/2} = \exp\left[\frac{\hbar}{2i}\left(\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}\right)\right]$$

$$-\frac{i}{\hbar}\sigma\tau' = -\frac{\hbar}{i}\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}$$

$$\boxed{F(q,p) = A(q,p) \; e^{\hbar\Delta/2i} \; B(q,p)}$$

Now we can write the dynamics of $\hat{\rho}$:

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[H, \hat{\rho}] \qquad \text{Liouville equation}$$

$$\dot{\rho}(q,p) \overset{\hbar \to 0}{=} -\frac{i}{\hbar}\left[H(q,p)\, e^{\hbar\Delta/2i}\, \rho(q,p) - \rho(q,p)\, e^{\hbar\Delta/2i}\, H(q,p)\right]$$

$$= -\frac{1}{2}\left[H \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\rho - H\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}\rho - \rho\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}H + \rho\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}H\right]$$

$$= \frac{\partial\rho}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial \rho}{\partial q}\frac{\partial H}{\partial p} \quad \rightsquigarrow \quad \text{Liouville Equation} \quad \left(\begin{array}{l}\text{the rest are} \\ \text{quantum} \\ \text{corrections}\end{array}\right)$$

The relation between ρ and f is $f = \frac{1}{2\pi\hbar}\rho$ but, due to the presence of ρ on both sides of the equation

$$\frac{\partial f}{\partial t} = -\frac{i}{\hbar}\left[H e^{\hbar\Lambda/2i} f - f e^{\hbar\Lambda/2i} H\right]$$

$$A \Lambda B = -B \Lambda A$$ due to the form of Λ

proof
$$A\Lambda B = \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} = -\frac{\partial B}{\partial p}\frac{\partial A}{\partial q} + \frac{\partial B}{\partial q}\frac{\partial A}{\partial p} = -B\Lambda A$$ ∎

$$\Rightarrow \quad \frac{\partial f}{\partial t} = -\frac{(2i)i}{\hbar} H \left(\frac{e^{-i\hbar\Lambda/2} - e^{+i\hbar\Lambda/2}}{2i}\right) f$$

$$= -\frac{2}{\hbar} H \sin\left(\frac{\hbar\Lambda}{2}\right) f$$

$$\hbar \to \quad \boxed{\frac{\partial f}{\partial t} + \{f, H\} = 0}$$ Liouville evolution

The first quantum correction

$$\frac{\partial f}{\partial t} = -\frac{\cancel{2}}{\cancel{\hbar}} H \frac{\cancel{\hbar}\Lambda}{\cancel{2}} f + \frac{2}{\hbar} H \frac{1}{3!}\left(\frac{\hbar\Lambda}{2}\right)^3 f$$

$$= -H\Lambda f + \frac{\hbar^2}{24} H\Lambda^3 f$$

$$H\Lambda^3 f = \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\Lambda^2 f - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}\Lambda^2 f =$$

$$= \frac{\partial H}{\partial p}\Lambda^2\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\Lambda^2\frac{\partial f}{\partial p} =$$

$$= \frac{\partial^2 H}{\partial p^2}\Lambda\frac{\partial^2 f}{\partial q^2} - \frac{\partial^2 H}{\partial q\partial p}\Lambda\frac{\partial^2 f}{\partial q\partial p} - \frac{\partial^2 H}{\partial q\partial p}\Lambda\frac{\partial^2 f}{\partial q\partial p} + \frac{\partial^2 H}{\partial q^2}\Lambda\frac{\partial^2 f}{\partial p^2}$$

For a typical Hamiltonian $H = \frac{p^2}{2m} + V(q)$

$$\cancel{\frac{\partial^3 H}{\partial p^3}\frac{\partial^2 f}{\partial q^3}} - \frac{\partial^3 H}{\partial q^3}\frac{\partial^3 f}{\partial p^3}$$

$$\frac{\partial f}{\partial t} = -\{f, H\} - \frac{\hbar^2}{24}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 f}{\partial p^3} \qquad (*)$$

From this equation we can read out the Ehrenfest theorem that states that the expectation values of a quantum observable evolve according to classical equation of motion if the potential is at most harmonic. In reality equation (*) says more, namely that the entire distribution evolves classically. The subtle part that still distinguish (*) from a classical equation relies in fact into the initial condition. The "size" of the Wigner distribution cannot be smaller than $\hbar$.

Exercises:

- Prove that the Wigner distribution for the nth state of the harmonic oscillator has the form
$$f_n(q,p) = \frac{(-1)^n}{(\pi\hbar)} e^{-2H/\hbar\omega} L_n(4H/\hbar\omega)$$
- Sketch the results (positions of the zeros and regions in (q,p) where $f>0$ $f<0$ for $n=0, 1, 2$.
- Calculate the Wigner distribution of a coherent states and its time evolution. Compare the result with the classical case.

ELECTRONS IN A MAGNETIC FIELD

- The classical result: Drude conductivity with magnetic field

$$\left[\frac{d\vec{p}}{dt}\right]_{scattering} = \left[\frac{dp}{\partial t}\right]_{field}$$

$$\frac{m\vec{v}_d}{\tau_m} = e\left[\vec{E} + \vec{v}_d \times \vec{B}\right]$$

Written in components for a 2DEG $\vec{B} = B\hat{e}_z$

$$\frac{m\left(v_x\hat{e}_x + v_y\hat{e}_y\right)}{\tau_m} = e\left[E_x\hat{e}_x + E_y\hat{e}_y + \left(v_x\hat{e}_x + v_y\hat{e}_y\right)\times B\hat{e}_z\right]$$

$$= e\left[E_x\hat{e}_x + E_y\hat{e}_y - v_x B\hat{e}_y + v_y B\hat{e}_x\right]$$

By projecting

$$E_x = \frac{mv_x}{e\tau_m} - v_y B$$

$$E_y = \frac{mv_y}{e\tau_m} + v_x B$$

$$\Rightarrow \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \frac{m}{e\tau_m} & -B \\ B & \frac{m}{e\tau_m} \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\vec{J} = e\vec{v}n$$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \frac{m}{e^2\tau_m n} & -\frac{B}{en} \\ \frac{B}{en} & \frac{m}{e^2\tau_m n} \end{pmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

$$= \sigma^{-1}\begin{pmatrix} 1 & -\mu B \\ \mu B & 1 \end{pmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix} \qquad \sigma = |e| n\mu \qquad \mu = \frac{|e|\tau_m}{m}$$

$$\bar{\bar{\rho}} = \bar{\bar{\sigma}}^{-1}$$

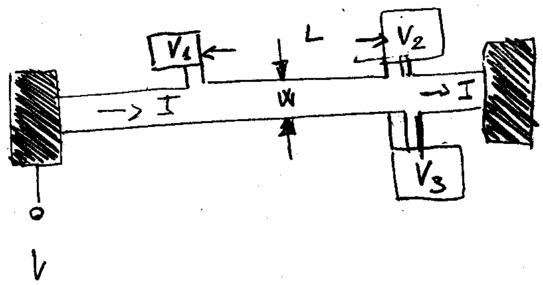
$$\begin{cases} \rho_{xx} = \sigma^{-1} & \text{- constant} \\ \rho_{xy} = -\dfrac{\mu B}{\sigma} = \dfrac{B}{en} = -\rho_{yx} & \text{- increase linearly with B} \end{cases}$$

Experimentally

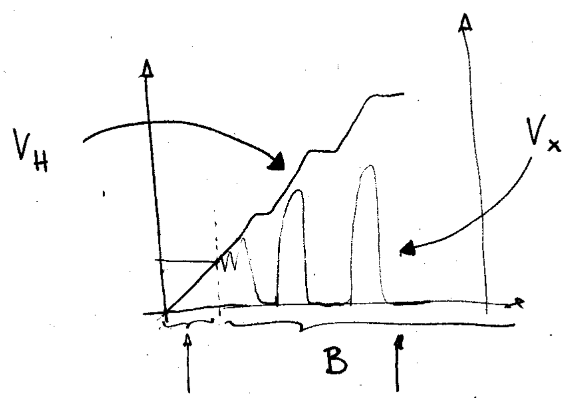
$\boxed{A}$ set up a longitudinal current

$\boxed{B}$ measure a $\left\langle \begin{array}{l} \text{longitudinal} \\ \text{transverse} \end{array} \right. \Big| \rightarrow$ voltage drop.

$$V_x = V_1 - V_2$$
$$V_H = V_2 - V_3$$



$$\rho_{xx} = \frac{V_x}{I} \cdot \frac{W}{L} \qquad \rho_{yx} = \frac{V_H}{I}$$



N.B. $$n = \left[|e| \frac{d\rho_{yx}}{dB} \right]^{-1} = \frac{\frac{I}{|e|}}{\frac{dV_H}{dB}}$$

$$\mu = \frac{1}{|e| n \rho_{xx}} = \frac{I/|e|}{n V_x \frac{W}{L}}$$

■ The quantum regime: Shubnikov-deHaas oscillations and Quantum Hall Effect.
Signatures of the Landau levels

The main point is that the density of states changes

$$N(E) = \frac{m}{\pi \hbar^2} \vartheta(E - E_s)$$

$$\downarrow$$

$$N(E,B) \approx \frac{2eB}{h} \sum_{n=0}^{\infty} \delta\left(E - E_s - \left(n+\tfrac{1}{2}\right)\hbar\omega_c\right) \qquad \omega_c = \frac{eB}{m}$$

The symbol $\approx$ is due to the fact that the density of states is actually not exactly a sum of δ functions due to the impurity potential that renormalizes (=> broadens) the position of the LANDAU LEVELS so are called the quantum mechanical eigenstates responsible for the change of the density of states.

Phenomenologically the resistivity has a peak any time the Fermi energy lies in the middle of a Landau band. This is due to the fact that a Landau level (unperturbed) carries no current. These unperturbed Landau levels are the only ones that energetically could carry the current when E_F is in the middle of a Landau band.

LL : euristic presentation

$$m\dot{v} = e\vec{v}\times\vec{B}$$

$$\dot{v}_x = -\frac{eB}{m} v_y \qquad \ddot{v}_x = -\left(\frac{eB}{m}\right)^2 v_x$$

$$\dot{v}_y = \frac{eB}{m} v_x \qquad \ddot{v}_y = -\left(\frac{eB}{m}\right)^2 v_y$$

$$v_x(t) = v_0 \cos(\omega_c t + \varphi)$$

$$v_y(t) = v_0 \sin(\omega_c t + \varphi)$$

v_0, φ are arbitrary in the classical (without radiation) motion.

$$r_c = \frac{v_0}{\omega_c}$$

Quantum mechanically the action is quantized

$$S_{cl} = \underbrace{\pi m v_0 r_c}_{\text{classical action}} = nh \qquad S_{cl} = \int_{orbit} p\,dx$$

$$2\pi m v_0 r_c = 2nh$$

$$\frac{m v_0^2}{\omega_c} = 2n\hbar \qquad \frac{1}{2} m v_0^2 = n\hbar\omega_c$$

$$E = E_s + n\hbar\omega_c$$

Notice that these kind of arguments do not give the zero point correction.

About the prefactor of the density of states: it comes from the analysis that all the states that before the application of the magnetic field were evenly distributed now should come all together at the energy $E_n = E_s + \left(n+\frac{1}{2}\right)\hbar\omega_c$

$$\frac{N_0}{L^2} = \frac{m}{\pi\hbar^2} \cdot \hbar\omega_c = \frac{\cancel{m}}{\pi\cancel{\hbar^2}} \cdot \frac{\cancel{\hbar} eB}{\cancel{m}} = \frac{2eB}{h} .$$

Which is the value of B at which the Landau levels can be seen?

$\omega_c^{-1} \ll \tau_m \quad \rightsquigarrow \quad$ a few classical orbits before scattering takes place

$\hbar\omega_c \gg \dfrac{\hbar}{\tau_m} \quad \rightsquigarrow \quad$ distance in energy between the LL larger than scattering induced broadening

random impurity potential which deform the LL.

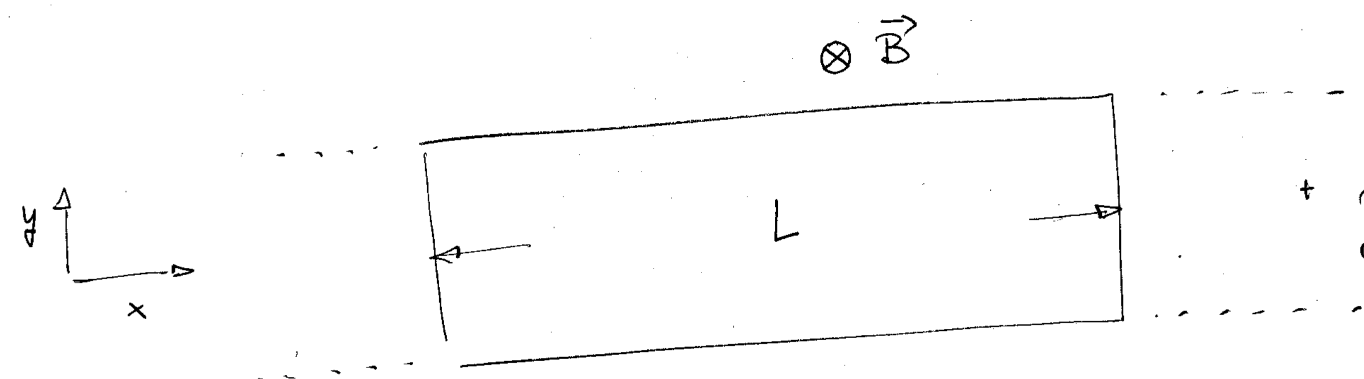
Landau Levels : a more rigorous treatment

Due to their similarities we will treat these 3 cases:

A - Confined electrons in zero magnetic field
B - Unconfined electrons in non-zero magnetic field
C - Confined electrons in non-zero magnetic field

The states that we will obtain in the 3 cases are

TRANSVERSE MODES (or MAGNETO-ELECTRIC SUBBANDS)



+ periodic boundary conditions in x.

$$\left[E_s + \frac{(-i\hbar\vec{\nabla} + e\vec{A})^2}{2m} + U(y)\right]\Psi(x,y) = E\,\Psi(x,y)$$

$$\vec{A} = -\hat{e}_x B y \quad \Rightarrow \quad A_x = -By \quad A_y = 0$$

$$\left[E_s + \frac{(p_x + eBy)^2}{2m} + \frac{p_y^2}{2m} + U(y)\right]\Psi(x,y) = E\,\Psi(x,y)$$

Due to translational invariance in the x direction:

$$\Psi(x,y) = \frac{1}{\sqrt{L}}\exp[ikx]\,\chi(y)$$

For the transverse function $\chi(y)$ the SE looks like:

$$\left[E_s + \frac{(\hbar k + eBy)^2}{2m} + \frac{p_y^2}{2m} + U(y)\right]\chi(y) = E\,\chi(y)$$

$\boxed{A}$ The equation for the transverse mode looks like

ELECTRIC SUBBANDS

$$\left[E_s + \frac{\hbar^2 k^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m\omega_0^2 y^2\right]\chi(y) = E\chi(y)$$

where we have taken for simplicity an harmonic confinement

$$U(y) = \frac{1}{2} m\omega_0^2 y^2$$

$$\chi_{n,k}(y) = u_n(q) \qquad q = \sqrt{\frac{m\omega_0}{\hbar}}\, y \qquad u_n(q) = e^{-q^2/2} H_n(q)$$

$$E(n,k) = E_s + \frac{\hbar^2 k^2}{2m} + \left(n+\frac{1}{2}\right)\hbar\omega_0 \qquad n = 0, 1, 2, \ldots$$

$$H_0(q) = \frac{1}{\sqrt[4]{\pi}} \qquad H_1(q) = \frac{\sqrt{2}}{\sqrt[4]{\pi}}\, q \qquad H_2(q) = \frac{2q^2-1}{\sqrt{2}\sqrt[4]{\pi}}$$

$$v(n,k) = \frac{1}{\hbar}\frac{\partial E(n,k)}{\partial k} = \frac{\hbar k}{m}$$

$\boxed{B}$

MAGNETIC SUBBANDS

$$\left[E_s + \frac{p_y^2}{2m} + \frac{(eBy + \hbar k)^2}{2m}\right]\chi(y) = E\chi(y)$$

The situation is not so much different from before: since k is a constant for the equation of the transverse subbands:

$$\frac{(eBy)^2}{2m} = \frac{1}{2} m\omega_c^2 y^2 \qquad \hbar k = eBy_k \qquad y_k = \frac{\hbar k}{eB} \qquad \omega_c = \frac{|e|B}{m}$$

$$\left[E_s + \frac{p_y^2}{2m} + \frac{1}{2} m\omega_c^2 (y+y_k)^2\right]\chi(y) = E\chi(y)$$

Now nevertheless the center of the subbands for each k is different.

$$v(n,k) = \frac{1}{\hbar}\frac{\partial E(n,k)}{\partial k} = 0$$

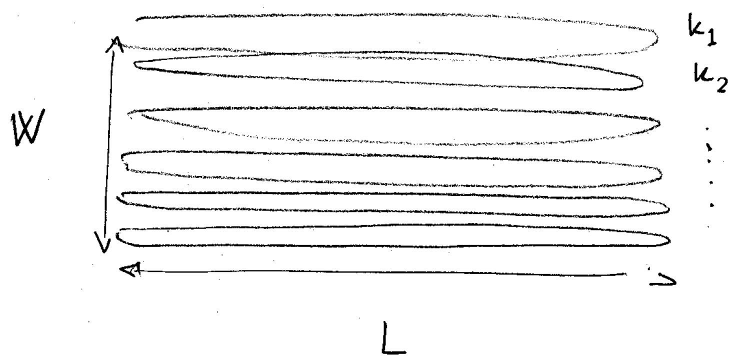
$$E(n,k) = E_s + \left(n+\frac{1}{2}\right)\hbar\omega_c$$

$$\psi_{kn}(x,y) = e^{ikx}\, u_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(y + \frac{\hbar k}{eB}\right)\right)$$

A wave packet localized in $\bar{x}$ would not move (Exercise). Spacial extent in the y direction is (this is the one of the lowest mode)

$$\Delta y \approx \sqrt{\frac{\hbar}{m\omega_c}} = \frac{1}{\omega_c}\sqrt{\frac{\hbar\omega_c}{m}} = \frac{v_0}{\omega_c} = x_0$$

v_0 is the equivalent max velocity of a classical oscillator of energy $\frac{\hbar\omega_c}{2}$.



How many electrons fits into a Landau level?

$$\Delta k = \frac{2\pi}{L} \qquad \Delta y_k = \frac{\hbar\,\Delta k}{|e|B} = \frac{2\pi\hbar}{|e|BL} \qquad N = 2 \times \frac{W}{\Delta y_k} = \frac{|e|BS}{\pi\hbar}$$

$\Rightarrow$ n (remember that the density of states is always written per unit "volume")

$$\boxed{n = \frac{2|e|B}{h}}$$

[C] Confined electrons in non-zero magnetic field

MAGNETO-ELECTRIC SUBBANDS

$$\left[E_s + \frac{p_y^2}{2m} + \frac{(eBy + \hbar k)^2}{2m} + \frac{1}{2} m\omega_0^2 y^2 \right] \chi(y) = E\chi(y)$$

$$\left(E_s + \frac{p_y^2}{2m} + \frac{1}{2} m \frac{\omega_0^2 \omega_c^2}{\omega_0^2 + \omega_c^2} y_k^2 + \frac{1}{2} m(\omega_c^2 + \omega_0^2)\left(y + \frac{\omega_c^2}{\omega_0^2 + \omega_0^2} y_k\right)^2 \right)$$

$$\chi(y) = E\chi(y)$$

Exercise?

proof $$\frac{(eBy + \hbar k)^2}{2m} = \frac{1}{2} m\omega_c^2 (y + y_k)^2$$

$$\frac{1}{2} m\omega_0^2 y^2 + \frac{1}{2} m\omega_c^2 (y + y_k)^2 = \frac{1}{2} m(\omega_c^2 + \omega_0^2)(y + A)^2 + B$$

$$\frac{1}{2} m(\omega_0^2 + \omega_c^2) y^2 + m\omega_c^2 y y_k + \frac{1}{2} m\omega_c^2 y_k^2 = \frac{1}{2} m(\omega_c^2 + \omega_0^2) y^2$$

$$+ \frac{1}{2} m(\omega_c^2 + \omega_0^2) A^2 + B$$

$$+ m(\omega_c^2 + \omega_0^2) A y$$

Setting the coefficients of the monomers equal to each other

$$A = \frac{\omega_c^2}{\omega_c^2 + \omega_0^2} y_k$$

$$B = \frac{1}{2} m \left[\omega_c^2 y_k^2 - \cancel{(\omega_c^2 + \omega_0^2)} \frac{\omega_c^4}{(\omega_c^2 + \omega_0^2)^{\cancel{2}}} y_k^2 \right]$$

$$= \frac{1}{2} m y_k^2 \omega_c^2 \left(1 - \frac{\omega_c^2}{\omega_c^2 + \omega_0^2} \right) = \frac{1}{2} m y_k^2 \frac{\omega_0^2 \omega_c^2}{\omega_c^2 + \omega_0^2}$$

$$\chi_{n,k}(y) = u_n\left(q + \frac{\omega_c^2}{\omega_c^2 + \omega_0^2} q_k\right) \qquad q = \sqrt{\frac{m\sqrt{\omega_0^2 + \omega_c^2}}{\hbar}}\, y$$

$$q_k = \sqrt{\frac{m\sqrt{\omega_0^2 + \omega_c^2}}{\hbar}}\, y_k$$

$$E(n,k) = E_s + \frac{1}{2} m \frac{\omega_0^2 \omega_c^2}{\omega_0^2 + \omega_c^2} y_k^2 + \left(n + \frac{1}{2}\right) \hbar \sqrt{\omega_0^2 + \omega_c^2}$$

$$= E_s + \left(n + \frac{1}{2}\right) \hbar \sqrt{\omega_0^2 + \omega_c^2} + \frac{\hbar^2 k^2}{2m} \frac{\omega_0^2}{\omega_c^2 + \omega_0^2}$$

$$v(n,k) = \frac{1}{\hbar} \frac{\partial E}{\partial k} = \frac{\hbar k}{m} \frac{\omega_0^2}{\omega_c^2 + \omega_0^2}$$

It seems that the only effect of the magnetic field is to renormalize the effective mass

$$m_{eff} = m \left(1 + \frac{\omega_c^2}{\omega_0^2} \right)$$

But the effect is more dramatic if we consider the special location of the carriers:

$$y_k = \frac{\hbar k}{eB} = \frac{\hbar k}{m} \cdot \frac{m}{eB} \cdot \frac{\omega_0^2}{\omega_c^2 + \omega_0^2} \cdot \frac{\omega_c^2 + \omega_0^3}{\omega_0^2}$$

$$= \frac{\hbar k}{m_{eff}} \frac{m_{eff}}{eB} = v(n,k) \cdot \frac{\omega_0^2 + \omega_c^2}{\omega_c \omega_0^2}$$