Quantentheorie II

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Blatt 11

1. Conserved quantities in the Dirac theory

• The Hamilton operator in the theory of Dirac is given by the expression $H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$ where β and α_k are the matrices

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

and σ_k are the Pauli matrices. Calculate the commutator of H with the operators

$$p^2 \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \qquad \vec{p}\cdot \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \qquad L^2 \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix},$$

where $\mathbb{I}$ is the 2×2 identity matrix, $\vec{L} = \vec{r}\times\vec{p}$ is the vector of the angular momentum operator and $\vec{\sigma}$ is the vector of the Pauli matrices. Which of these operators is then a conserved quantity in the theory of Dirac? (**4 Points**)

2. Time dependent perturbation theory

Consider a two level atom with the states $|1\rangle$ and $|2\rangle$ in presence of a time dependent perturbation. The Hamilton operator of the system can be written as:

$$H(t) = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + V(t)(|1\rangle\langle 2| + |2\rangle\langle 1|)$$

with $V(t) = V_0 e^{-t^2/\tau^2}\cos\omega t$.

a) • Calculate with the help of the time dependent perturbation theory (for small V_0) the probability to find the system in the state $|2\rangle$ in the limit $t\to\infty$ knowing that at the beginning ($t\to-\infty$) the system was in the state $|1\rangle$. (**2 Points**)

b) Calculate the approximate transition probability in the case $\hbar\omega = E_2 - E_1$ and $\hbar/\tau = 0.1(E_2 - E_1)$?

3. Variational method

Consider two identical bosons of mass m confined to a one dimensional harmonic potential with frequency ω. The interaction between the bosons is of the form $U(x_1, x_2) = U_0\delta(x_1 - x_2)$ with $U_0 > 0$.

a) • Show that a variational Ansatz for the ground state that associates to each the two bosons the wave function:

$$\phi_\lambda(x) = \frac{1}{\sqrt{\sqrt{\pi}\lambda}}\exp\left(-\frac{x^2}{2\lambda^2}\right)$$

leads to an equation for the optimal parameter λ of the form

$$\frac{\hbar^2}{m\lambda^2} - m\omega^2\lambda^2 + \frac{U_0}{\sqrt{2\pi}\lambda} = 0.$$

(**4 Points**)

b) Consider the limit of weak interaction $\frac{U_0}{x_0 \hbar\omega} \ll 1$ and prove that in this limit the optimal lambda can be approximated with $\lambda \approx x_0[1+\frac{A}{4}+\mathcal{O}(A^2)]$ where $A = \frac{U_0}{\sqrt{2\pi} x_0 \hbar\omega}$ and $x_0 = \sqrt{\frac{\hbar}{m\omega}}$.

Frohes Schaffen!

Return the solution of the exercises marked with • by Monday 19th of January at 10:00.