Quantentheorie II

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Blatt 6

1. Classical vs. relativistic fermions

The density of states of a generic system of particles can be expressed as the sum over Dirac delta functions centered on the single particle eigenenergies, namely:

$$\rho(E) = \sum_{\{\alpha\}} \delta(E - E_\alpha)$$

where $\{\alpha\}$ represent a complete set of quantum numbers for the system.

a) • Calculate the density of states for a set of N fermions confined to a cubic region of volume V. How would the result change for a confinement to a segment of length L or a square of surface S? Can you imagine a physical realization of such confinement to reduced dimensions for electrons? *Hint*: Perform all calculations of this exercise in the thermodynamic limit: *i.e.* $N \to \infty$ and V (or, correspondingly S or L) $\to \infty$ but maintaining a constant density. (**2 Points**)

b) Repeat the calculation performed at point *a*) but this time consider a set of relativistic electrons whose kinetic energy is described by the equation

$$E_K = \sqrt{m_0^2c^4 + p^2c^2} - m_0c^2.$$

c) • Calculate the Fermi energy for all the systems described at point a) and b). *Hint*: For what concerns the relativistic case remember that $E_F \ll m_0c^2$ or the system becomes unstable and the number of particles cannot be fixed. (**2 Points**)

2. Helium atom

The Hamiltonian for two electrons that move in the Coulomb potential generated by a nucleus of charge Z is $H = H_0 + H_1$ with

$$H_0 = -\frac{\hbar^2}{2\mu}(\triangle_1 + \triangle_2) - Ze^2\left(\frac{1}{|\vec{x}_1|} + \frac{1}{|\vec{x}_2|}\right)$$

and

$$H_1 = \frac{e^2}{|\vec{x}_1 - \vec{x}_2|}\,.$$

a) Calculate the ground state energy to first order in the perturbation theory assuming that the 0th order state is

$$\psi_0(\vec{x}_1, \vec{x}_2) = \frac{Z^3}{\pi a_0^3} \exp\left[-\frac{Z(|\vec{x}_1| + |\vec{x}_2|)}{a_0}\right]$$

Hint: You may use the expansion in multipoles:

$$\frac{1}{|\vec{x}_1 - \vec{x}_2|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta_1, \phi_1) Y_{\ell m}(\theta_2, \phi_2) \,,$$

where $r_< = \min(|\vec{x}_1|, |\vec{x}_2|)$ and $r_> = \max(|\vec{x}_1|, |\vec{x}_2|)$ and $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics.

b) Calculate the ground state energy also by minimizing the functional $E(\beta) = \langle\psi_\beta| H |\psi_\beta\rangle$ within the family of two electron functions

$$\psi_\beta(\vec{x}_1, \vec{x}_2) = \frac{\beta^3}{\pi a_0^3} \exp\left[-\frac{\beta(|\vec{x}_1| + |\vec{x}_2|)}{a_0}\right] .$$

Compare the two results for different values of $Z (Z = 1, \dots, 6)$.

3. Electrons in a parabolic potential

Consider a system of non-interacting electrons vertically confined to a plane and under the influence of the two-dimensional parabolic potential $V(x, y) = \frac{1}{2} m\omega^2 (x^2 + y^2)$

a) • Calculate, for the case $N \gg 1$ the dependence of the total energy on the total number of electrons N. **(2 Points)**

b) • Calculate the five lowest "magic numbers" for the harmonic oscillator, *i.e.* the particle numbers $N_1, N_2, \dots$ at which, in the ground state, one single particle energy level of the harmonic oscillator is completely full and the next empty. *Hint*: Notice that the single particle energy levels can be orbitally degenerate. **(2 Points)**

c) • Write explicitly the wave function corresponding to the ground states for two and three particles. **(2 Points)**

Frohes Schaffen!

Return the solution of the exercises marked with • by Monday 24th of November at 10:00.