Quantentheorie II

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Blatt 4

1. Spontaneous decay in a hydrogen atom

• The rate of spontaneous decay of the state $|\psi_i\rangle$ into the state $|\psi_f\rangle$ is given by the formula

$$\Gamma_{if} = \frac{\alpha\omega_{if}^3}{2\pi c^2}\int \mathrm{d}\Omega_k \sum_{j=1}^{2} |\langle\psi_f|\, \vec{e}_j(\vec{k})\cdot\vec{r}\,|\psi_i\rangle|^2 \,,$$

where α is the fine structure constant, c is the speed of light and ω the transition frequency from $|\psi_i\rangle$ to $|\psi_f\rangle$. $\vec{e_1}(\vec{k})$ and $\vec{e_2}(\vec{k})$ are the unitary vectors pointing in the the two different polarization direction of a photon of wavevector $\vec{k}$. The integral must be performed over the complete solid angle Ω_k of $\vec{k}$. Calculate the decay rate of the first excited p-state ($n = 2$, $l = 1$, $m = 0$) into the ground state of hydrogen. (**3 Points**)

2. Selection rules

• Consider the matrix elements

$$\langle n'l'm_l'm_s'|\, (3z^2 - r^2)\, |nlm_lm_s\rangle \,,$$

$$\langle n'l'm_l'm_s'|\, xy\, |nlm_lm_s\rangle$$

of a one-electron atom. Write, with the help of the composition properties and orthogonality of the spherical harmonics, the selection rules for Δl and Δm_l. Why is $\Delta m_s = 0$ in both cases? (**2 Points**)

3. Coherent states of a harmonic oscillator

There is a simple correspondence between coherent states and complex numbers: to every complex number α one can associate the coherent state:

$$|\alpha\rangle = D(\alpha)\,|0\rangle$$

where $|0\rangle$ is the ground state of the harmonic oscillator and $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$, $a^\dagger$ and a being the creation and annihilation operators defined by the relations:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad \text{and} \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)\,.$$

a) • Prove that each coherent state is a superposition of all eigenstates of the harmonic oscillator by demonstrating that

$$\langle n|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\frac{\alpha^n}{\sqrt{n!}}$$

where $|n\rangle$ is the n-th eigenstate of the harmonic oscillator. *Hint*: You can use the Glauber formula

$$\mathrm{e}^{A+B} = \mathrm{e}^A\mathrm{e}^B\mathrm{e}^{-\frac{1}{2}[A,B]}$$

valid for two operators A and B which commute with their commutator. (**2 Points**)

b) Show with the help of point a) that:

$$|\alpha\rangle = \exp\left(-i\frac{p_\alpha x_\alpha}{2\hbar}\right)\exp\left(i\frac{p_\alpha \hat{x}}{\hbar}\right)\exp\left(-i\frac{x_\alpha \hat{p}}{\hbar}\right)|0\rangle$$

with $x_\alpha := \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}(\alpha)$ and $p_\alpha := \sqrt{2\hbar m\omega}\,\mathrm{Im}(\alpha)$. Write explicitly the wave function $\psi_\alpha(x) \equiv \langle x|\alpha\rangle$ and sketch the probability density $|\psi_\alpha(x)|^2$ of the coherent state corresponding to $\alpha = x_0 \in \mathbb{R}$.

c) • Assume to have prepared a harmonic oscillator at time $t = 0$ in a coherent state $|\alpha_0\rangle$. Prove that at time t it will be evolved into the state $\exp(-i\omega t/2)\,|\alpha_0 \mathrm{e}^{-i\omega t}\rangle$ **(3 Points)**

d) At time $t = 0$ the state of the harmonic oscillator is described by the superposition of 2 coherent states

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(\,|x_0\rangle + |-x_0\rangle)$$

with $x_0 \in \mathbb{R}$. Calculate the evolution of the state at every time and sketch $|\psi(x,t)|^2$ for $\omega t = 0,\ \pi/2$ and π.

Frohes Schaffen!

Return the solution of the exercises marked with • by Monday 10th of November at 10:00.