Quantentheorie II

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Blatt 2

1. WKB method

Let us consider the one dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

with a potential $V(x)$ that has only one local minimum with respect to x. We indicate with a and b the turning points of the classical trajectory at energy E. They are defined by the relation $V(a) = V(b) = E$. According to the semiclassical WKB-Ansatz, the eigenfunction $\psi(x)$ of energy E is, in the classically allowed region $a < x < b$, the linear combination of the functions:

$$\phi_\pm(x) = \frac{1}{\sqrt{p(x)}}\exp\left[\pm\frac{i}{\hbar}\int_{x_0}^{x} p(x')dx'\right], \quad \text{with} \quad p(x) = \sqrt{2m[E - V(x)]},$$

where $x_0 \in [a, b]$. The condition that $\psi(x)$ does not diverge in the forbidden regions implies

$$\psi(x) = \frac{2}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_a^x p(x')dx' - \frac{\pi}{4}\right], \quad \text{as well as} \tag{1}$$

$$\psi(x) = \frac{2C}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_x^b p(x')dx' - \frac{\pi}{4}\right], \tag{2}$$

where C is in general a complex amplitude.

a) Show that equations (1) and (2) are satisfied only if

$$\int_a^b \sqrt{2m[E - V(x)]}dx = \pi\hbar(n + 1/2),$$

with $n \in \mathbb{N}_0$ holds and correspondingly $C = (-1)^n$.

b) • In a potential with a hard (left) wall in $x = 0$, $a = 0$ for all energies and the condition (1) is substituted by the boundary condition $\psi(0) = 0$. Prove that, in this case, the relation

$$\int_0^b \sqrt{2m[E - V(x)]}dx = \pi\hbar(n + 3/4)$$

holds, with $n \in \mathbb{N}_0$. **(3 Points)**

c) • Calculate with this formula the semiclassical eigenenergies for the potential

$$V(x) = \begin{cases} \infty & : x < 0 \\ \alpha x^2 & : x \geq 0 \end{cases},$$

with $\alpha > 0$. Prove that for $\alpha = \frac{m\omega^2}{2}$ you find a spacing between the energy levels of $2\hbar\omega$. Can you explain the result using the symmetry of the eigenstates of the harmonic oscillator? **(3 Points)**

2. Baker-Hausdorff equation

The exponential of a linear operator A is defined by the relation:

$$\mathrm{e}^A = \sum_{m=0}^{\infty} \frac{A^m}{m!}.$$

a) • Prove the so called Baker-Campbell-Hausdorff formula for two operators A, B

$$\mathrm{e}^A B \mathrm{e}^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!} [A, B]_m,$$

where $[A, B]_m = [A, [A, B]_{m-1}]$ and $[A, B]_0 = B$.
Hint: Construct the Taylor expansion of the auxiliary function $f(s) \equiv \mathrm{e}^{sA} B \mathrm{e}^{-sA}$ $(s \in \mathbb{R})$ centered in $s = 0$. For this purpose you will need the derivative of the exponential of an operator: $(\mathrm{d}/\mathrm{d}s)\mathrm{e}^{\pm sA} = \pm A \mathrm{e}^{\pm sA}$. Eventually, evaluate the expansion in the point $s = 1$ in order to obtain $\mathrm{e}^A B \mathrm{e}^{-A}$. **(4 Points)**

b) Show that, in the case $[A, [A, B]] = [B, [B, A]] = 0$ the following formula holds:

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^B \mathrm{e}^A \mathrm{e}^{[A,B]}.$$

This formula is also called Baker-Hausdorff formula and is useful for the calculation with creator and annihilator operators.

Frohes Schaffen!

Return the solution of the exercises marked with • by Monday 27th of October at 10:00.