Applications of Group Theory

Lectures	Tue	10:00 - 11:30	PHY 9.1.09
	Thu	10:00 - 11:30	PHY 9.1.09
Exercises	Fri	10:00 - 11:30	PHY 5.0.21

Sheet 11

1. Finite vs. infinitesimal rotations

Consider the Euler-Rodrigues parametrization of rotations which associates to the rotation of an angle $\phi \in [0, \pi]$ around the axis defined by the pole $\vec{n}$ the parameters $R(\phi, \vec{n}) \to R(\lambda, \vec{\Lambda}) := R(\cos\frac{\phi}{2}, \vec{n}\sin\frac{\phi}{2})$ as well as $R(\phi, \vec{n}) \to R(-\lambda, -\vec{\Lambda})$, together with the composition rule:

$$R(\lambda_1, \vec{\Lambda}_1)R(\lambda_2, \vec{\Lambda}_2) = R(\lambda_1\lambda_2 - \vec{\Lambda}_1 \cdot \vec{\Lambda}_2,\, \lambda_1\vec{\Lambda}_2 + \lambda_2\vec{\Lambda}_1 + \vec{\Lambda}_1 \times \vec{\Lambda}_2).$$

1. Prove that the necessary and sufficient condition for two rotations to commute is either to be coaxial or bilateral binary.
   Hint: The second condition translates into the Euler-Rodriguez parameters $\lambda_1 = \lambda_2 = 0$ and $\vec{\Lambda}_1 \perp \vec{\Lambda}_2$.
2. Prove that two infinitesimal rotations always commute if the result is retained up to the lowest non vanishing order in the rotation angles.
3. Calculate the commutator of two infinitesimal rotations up to the second order in the rotation angles and prove the relation
$$[R(\lambda_1, \vec{\Lambda}_1), R(\lambda_2, \vec{\Lambda}_2)] = R(1, 2(\vec{\Lambda}_1 \times \vec{\Lambda}_2)) - R(1, \vec{0}) + o(\delta\phi^2)$$

2. Projective factors

Given a group of order h, the necessary and sufficient condition for the h^2 numbers $[g_i, g_j]$ to be a factor system is the associativity conditions:

$$[g_i, g_j][g_ig_j, g_k] = [g_i, g_jg_k][g_j, g_k].$$

1. By application of the associativity conditions, prove the following equalities for the projective factors involving the identity and the inverse:
$$[E, E] = [E, g] = [g, E]$$
   and
$$[g, g^{-1}] = [g^{-1}, g].$$
2. By explicit calculation by means of the quaternion representation of the rotations, prove that the projective factors for spinorial representations read:
$$[E, g] = 1, \qquad [g, g^{-1}] = 1 \quad \text{for} \quad g \neq C_2, \qquad [g, g^{-1}] = -1 \quad \text{for} \quad g = C_2$$

Frohes Schaffen!