Quantum theory of condensed matter I

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	Thu	10:00 - 12:00	H33
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Sheet 8

1. Compressibility of a 2D electron gas

1. Calculate the density of states (DoS) per unit volume $\nu(\epsilon)$ for spin-degenerate 2D electron gas having a parabolic spectrum, $\epsilon_k = \hbar^2 k^2/2m$. Make sure that two definitions, namely, $\nu(\epsilon) = \sum_{\mathbf{k}} \delta(\epsilon - \epsilon_k)$ and $g_s \sum_{\mathbf{k}} f(\mathbf{k}) = g_s \int \nu(\epsilon_k) \mathrm{d}\epsilon_k \int (\mathrm{d}\phi/2\pi) f(\mathbf{k})$, are equivalent. Note that it is frequently customary to deal with the DoS per spin projection, and to introduce the degeneracy factor g_s (in our case, $g_s = 2$) which tells how many electrons may occupy a given $\mathbf{k}$ state. **(1 Point)**

2. Considering a spin-degenerate 2D electron gas in thermal equilibrium characterized by temperature T and chemical potential μ, find the electron density $n(T, \mu)$ and compressibility $\chi(T, \mu) = \partial n/\partial\mu$. Using their general form as an integral over energy, find explicit expressions for 2 limits: (a) degenerate Fermi gas, $\mu \gg T$ and (b) non-degenerate Boltzmann gas, $|\mu| \gg T$, $\mu < 0$. **(2 Points)**

2. Kinetics in crossed electric and magnetic field

Consider 2D electron gas subject simultaneously to an infinitesimally weak static electric field $\mathbf{E}$ and to a static magnetic field $\mathbf{B}$, directed perpendicular to the 2D plane. Assuming diffusive transport in the plane and $T \ll \mu$ (as in the lecture), find the conductivity $\hat{\sigma}$ and resistivity $\hat{\rho} = \hat{\sigma}^{-1}$ tensors which are defined through $\mathbf{j} = \hat{\sigma}\mathbf{E}$ and $\mathbf{E} = \hat{\rho}\mathbf{j}$. Please proceed as follows.

2.1 Consider the case $\mathbf{E} = 0$ first. Analyse the term in Boltzmann equation containing the magnetic force and show that a perpendicular magnetic field alone does not modify the equilibrium distribution. Explain why. Use the notation ω_c for the cyclotron frequency eB/m. **(1 Point)**

2.2 Consider now very small $\mathbf{E} \neq 0$ and find the first-order correction $\delta f \propto |\mathbf{E}|$ to equilibrium distribution function. You can repeat the steps given in the lecture, but take into account that now there are two components of current, one parallel and one perpendicular to the electric field. In order to better understand the procedure, look for a solution in the form $\delta f = (-\partial f_0/\partial\epsilon_k) \sum_n g_n e^{in\phi}$ and consider the action of the collision integral on individual angular harmonics $\mathrm{St}\{e^{in\phi}\} = -\tau_n^{-1} e^{in\phi}$. Find the eigen values τ_n^{-1} of the collision operator. For $n = \pm 1$ they should reproduce the transport scattering rate τ^{-1} from the lecture. Find g_n up to 1st order in $\mathbf{E}$. Show that only g_n with $n = \pm 1$ (which enter the electrical current) are present in the perturbative expansion to that order. Explain why. **(3 Points)**

2.3 Express the electrical current in terms of $g_{\pm 1}$ and find the conductivity and resistivity tensors. Make sure that they obey the symmetry relations $\sigma_{xx}(-B) = \sigma_{xx}(B) = \sigma_{yy}(B)$ and $\sigma_{xy}(-B) = -\sigma_{xy}(B) = \sigma_{yx}(B)$. Express $\hat{\sigma}$ and resistivity $\hat{\rho}$ in terms of parameter $\omega_c\tau$ and $\sigma_0 = \sigma_{xx}|_{B=0}$ found in the lecture. Draw the resulting σ_{xx}, σ_{xy}, ρ_{xx}, and ρ_{xy} as a function of B and discuss limiting cases $\omega_c\tau \gg 1$ and $\omega_c\tau \ll 1$ (classically strong and weak B, correspondingly). **(2 Points)**

3. Classical Kubo formula

A classical analogue of the Kubo formula reads, $\hat{\sigma} = e^2\nu\hat{D}$, where the diffusion (or, diffusivity) tensor $D_{ml} = \int_0^\infty \mathrm{d}t \langle v_m(t) v_l(0)\rangle$, and angular brackets denote the average over ensemble. In our 2D case at $T \ll \mu$, this means average over initial directions of velocity for particles at the Fermi surface, $v_{x,y}(0) = v_F\{\cos\phi_0, \sin\phi_0\}$ and $\langle \ldots \rangle = \int (\mathrm{d}\phi_0/2\pi) \ldots$. Recall that diffusion describes current arising in the presence of a density gradient $\mathbf{j} = -e\hat{D}\partial n/\partial\mathbf{r}$ and is expressed through conductivity via the Einstein relation above.

Demonstrate that this approach reproduces the conductivity tensor obtained in Exercise 2. Consider distribution in the form $f = \delta(\epsilon_k - \epsilon_F) \sum_n g_n(t) e^{in\phi}$ and find solution to the Boltzmann equation $(\partial/\partial t + \omega_c \partial/\partial\phi) f = \mathrm{St}\{f\}$ with initial condition $f(t=0) = 2\pi\delta(\phi - \phi_0)\delta(\epsilon_k - \epsilon_F)$. Note that this equation describes ensemble-averaged stochastic time evolution of the initial distribution in the absence of electric field but in the presence of magnetic field. Using results from exercises 2.1 and 2.2, find solutions for $g_n(t)$ as well as averages $\langle v_m(t) v_l(0)\rangle \equiv M_{ml}(t) v_F^2/2$. Show that, in the presence of $B \neq 0$, the memory function $\hat{M}(t)$ has components $M_{xx} = M_{yy} = e^{-t/\tau}\cos\omega_c t$ and $M_{xy} = -M_{yx} = e^{-t/\tau}\sin\omega_c t$. Using $\hat{M}(t)$, derive expression for the dynamic conductivity $\hat{\sigma}(\omega)$ in the magnetic field. In the static limit, it should reproduce results from exercise 2, and in the limit $B \to 0$ – the result for $\sigma_{xx}(\omega)$ from the lecture.

Frohes Schaffen!