Quantum Theory of Condensed Matter I

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Wednesdays 16:15

Sheet 7

1. A fermionic two orbital system

Let us consider the physical system consisting of fermions allowed to occupy two orbitals. The Hamiltonian is given by:

$$H = \varepsilon_1 c_1^\dagger c_1 + \varepsilon_2 c_2^\dagger c_2 + b c_1^\dagger c_2 + b^* c_2^\dagger c_1. \tag{1}$$

In order to study the dynamics of the system we are interested in its retarded Green's function:

$$G^R_{nm}(t,t') = -\frac{i}{\hbar}\theta(t-t')\langle\{c_n(t), c_m^\dagger(t')\}\rangle, \tag{2}$$

where the creation and annihilation operators are in Heisenberg picture and the average is taken with respect to a thermal distribution.

1. Prove that the retarded Green's function only depends on the difference between the initial time t' and final time t.

$$G^R_{nm}(t,t') = G^R_{nm}(t-t')$$

(2 Points)

2. Calculate the equation of motion for the retarded Green's function in the frequency domain. In other terms, using (1) and the definition (2) prove that

$$(\hbar\omega + i\eta)G^R_{nm}(\omega) = \delta_{nm} + \sum_l H_{nl} G^R_{lm}(\omega)$$

or, in other terms that

$$\mathbf{G}^R(\omega) = [\hbar\omega + i\eta - \mathbf{H}]^{-1},$$

where η is an infinitesimal positive quantity ,we set without any loss of generality (thanks to the previous point) $t' = 0$ and finally $H_{nl} = \langle n|H|l\rangle$ are the matrix elements of H in the single particle basis $|n\rangle$ with $n = 1, 2$.

(2 Points)

3. Calculate the retarded Green's function in the time domain. In particular check explicitly the limit:

$$\lim_{t\to 0} G^R_{nm}(t) = -\frac{i}{\hbar}\delta_{nm}.$$

Hint: Make use of the Cauchy's residue theorem.

(2 Points)

4. The Green's functions have a probabilistic interpretation. Namely, if $P_{nm}(t)$ is the probability of finding the system at time t in the state n when it was prepared in m at time $t = 0$, then:

$$P_{nm}(t) = \left|\hbar G^R_{nm}(t)\right|^2 .$$

Calculate the probabilities $P_{11}(t)$ and $P_{21}(t)$ and prove that, as it is natural, $P_{11}(t) + P_{21}(t) = 1$ for all times. Make a plot of the result. For $b \neq 0$ the probabilities P_{nm} oscillate periodically in time. Why?

(2 Points)

5. Analyze the results obtained at the previous point. In particular consider the limits $\left|\frac{\varepsilon_1 - \varepsilon_2}{2b}\right|^2 \ll 1$ and $\left|\frac{\varepsilon_1 - \varepsilon_2}{2b}\right|^2 \gg 1$. Prove that in the first case all probabilities $P_{nm}(t)$ oscillate between 0 and 1 while in the second case the oscillation is very weak and $P_{nm}(t) \approx \delta_{nm}$ for all positive times. Comment the result.

(2 Points)

Frohes Schaffen!