Quantum Theory of Condensed Matter

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Sheet 9

1. The two spins problem

Consider a system of two spins described by a ferromagnetic Heisenberg Hamiltonian in an external homogeneous magnetic field $\boldsymbol{B}$.

$$H = -2J\boldsymbol{S}_1 \cdot \mathbf{S}_2 - g\mu_B \mathbf{B} \cdot (\mathbf{S}_1 + \mathbf{S}_2),$$

where $\mathbf{S}_i$ are spin-$\frac{1}{2}$ operators that, in second quantization, read $\mathbf{S}_i = \frac{\hbar}{2}\sum_{\tau\tau'} c^\dagger_{i\tau}\boldsymbol{\sigma}_{\tau\tau'}c_{i\tau'}$ with $\boldsymbol{\sigma}$ the vector of Pauli matrices.

a) Prove that the exchange part of the Hamiltonian can be written in the form

$$H_{\rm ex} = -J(S_1^+S_2^- + S_1^-S_2^+ + 2S_1^zS_2^z)$$

where $S_i^\pm$ are the usual raising (lowering) operators for the spin.

b) Prove the following useful relations:

$$\begin{aligned}
S_i^\mp S_i^\pm &= \frac{\hbar^2}{2} \mp \hbar S_i^z, \\
S_i^\pm S_i^z &= -S_i^z S_i^\pm = \mp\frac{\hbar}{2}S_i^\pm, \\
(S_i^+)^2 &= (S_i^-)^2 = 0, \\
(S_i^z)^2 &= \frac{\hbar^2}{4}.
\end{aligned}$$

c) Prove that the average magnetization for the system is independent from the site i: $\langle S_1^z\rangle = \langle S_2^z\rangle \equiv \langle S^z\rangle$.

d) Consider the retarded Green's function

$$G^r_{11}(t) = -i\theta(t)\langle[S_1^-(t), S_1^+]\rangle$$

and calculate it explicitly using the equation of motion technique. Prove that the result reads

$$G^r_{11}(\omega) = -\frac{\hbar^2\langle S^z\rangle}{\omega^+ - E_1} - \frac{\hbar}{2}\frac{\eta_+}{\omega^+ - E_2} - \frac{\hbar}{2}\frac{\eta_-}{\omega^+ - E_3}$$

where $\eta_\pm = \hbar\langle S^z\rangle \pm \langle S_1^+S_2^-\rangle + 2\langle S_1^zS_2^z\rangle$ while $E_1 = -\hbar b$, $E_2 = -\hbar b - 2J\hbar^2$ and $E_3 = -\hbar b + 2J\hbar^2$. We have introduced the parameter $b = \frac{1}{\hbar}g\mu_B B$.

e) Prove that the average magnetization for the system reads

$$\langle S^z\rangle = \frac{\hbar}{2}\frac{\exp(\beta\hbar b) - \exp(-\beta\hbar b)}{1 + \exp(\beta\hbar b) - \exp(-\beta\hbar b) + \exp(-2\beta\hbar^2 J)}$$

Hint: Make use of the spectral theorem and restrict first yourself to the case $b \neq 0$.

Frohes Schaffen!