Quantum Theory of Condensed Matter

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Sheet 8

1. Spectral density of states

In the previous two sheets we have make use of a fundamental relation that identifies in the spectral function the spectral density of states for a system, namely:

$$\langle n_\nu \rangle = \int \frac{d\omega}{2\pi} A_\nu(\omega) n_{F,B}(\omega), \tag{1}$$

where ν identifies a certain single particle state for the system, $\langle \bullet \rangle$ is the thermodynamical average, $n_{F,B}$ is the Fermi or Bose function depending on the nature of the particles that compose the system and $A_\nu(\omega)$ is the spectral function. In this sheet you will prove the relation (1) step by step. For this reason it is useful to introduce the following set of 4 Green's functions:

$$\begin{aligned}
G^r_\nu(t) &= -i\theta(t)\langle [c_\nu(t), c^\dagger_\nu]_\mp \rangle \\
G^a_\nu(t) &= +i\theta(-t)\langle [c_\nu(t), c^\dagger_\nu]_\mp \rangle \\
G^>_\nu(t) &= -i\langle c_\nu(t) c^\dagger_\nu \rangle \\
G^<_\nu(t) &= -i(\pm)\langle c^\dagger_\nu c_\nu(t) \rangle
\end{aligned}$$

where r stands for retarded, a for advanced and $[\bullet, \bullet]_\mp$ indicates the commutator or the anti-commutator depending on the statistics of the particles involved. When two signs are present the first one refers to the bosonic case while the second to the fermionic. Finally, the spectral function is defined as $A_\nu(\omega) \equiv -2\mathrm{Im}[G^r_\nu(\omega)]$.

a) Prove that the average density of ν particles $\langle n_\nu \rangle$ is related to the Fourier transform of the lesser Green's function $G^<_\nu(t)$ by the relation

$$\langle n_\nu \rangle = \pm \lim_{t\to 0} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} i G^<_\nu(\omega).$$

b) Prove that the following relation between the Green's functions holds at every time t:

$$i[G^>_\nu(t) - G^<_\nu(t)] = G^r_\nu(t) - G^a_\nu(t)$$

c) Prove that the relation $G^a_\nu(\omega) = G^r_\nu(\omega)^*$, where * indicates the complex conjugation, links the retarded to the advanced Green's functions for every ω .

d) Prove that the lesser and greater Green's functions are instead connected by the relation $G^<_\nu(\omega) = \pm e^{-\beta\omega} G^>_\nu(\omega)$. *Hint*: It can be useful to consider the imaginary time extension of the Heisenberg picture $c_\nu(i\beta) \equiv e^{-\beta H} c_\nu e^{\beta H}$.

e) By combining all the relation proven in the previous steps demonstrate the validity of the relation (1).

f) Check explicitly the relations proven at points b)-d) for a system of non-interacting particles (fermions and bosons) described by the Hamiltonian $H = \sum_\nu \epsilon_\nu c^\dagger_\nu c_\nu$.

Frohes Schaffen!