University of Regensburg Summer Semester 2009

Quantum Theory of Condensed Matter

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Mo 10:15, 5.0.21
We 16:15, H33

Sheet 2

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1. Bosonic commutation relations

Refresh the physics of the simple harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2},$$

which can be written in "second quantized" form, by expressing $\hat{x}$ and $\hat{p}$ in terms of *boson* creation and annihilation operators:

$$\hat{H} = \hbar\omega\left(b^\dagger b + \frac{1}{2}\right), \quad b^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\hat{x} - \mathrm{i}\frac{\hat{p}}{\sqrt{m\omega}}\right).$$

From the canonical commutation relations between position and momentum operators, it follows immediately (do you remember it?) that the basis commutation relations hold:

$$[b, b^\dagger] = 1, \quad [b, b] = 0, \quad b\,|0\rangle = 0$$

where $[A, B] = AB - BA$, $|0\rangle$ is the vacuum, and $\dagger$ indicates the Hilbert space adjoint.

a) Show that for two non commuting bosonic operators A, and B it holds

$$[A, B^n] = \sum_{k=0}^{n-1} B^k\,[A, B]\,B^{n-1-k}.$$

b) Prove, using the result of point a), the following relations valid for bosonic operators b, $b^\dagger$

$$\begin{aligned}\left[b, (b^\dagger)^n\right] &= n(b^\dagger)^{n-1} = \frac{\partial (b^\dagger)^n}{\partial b^\dagger},\\ \left[b^\dagger, b^n\right] &= -nb^{n-1} = -\frac{\partial b^n}{\partial b}.\end{aligned}$$

2. Classical vibration in the diatomic linear crystal

Consider a periodic chain of atoms with alternating masses m, and M connected by springs with spring constant K. The equilibrium position of the atoms is na with $n \in \mathbb{Z}$.

a) Prove that the dispersion relation of the eigenmodes reads:

$$\omega(q)^2 = \frac{K}{Mm}\left[M + m \pm \sqrt{M^2 + m^2 + 2mM\cos(2qa)}\right]$$

b) Make a sketch of the dispersion relation and discuss the dynamics of the two modes in the limiting cases $qa \to 0$ and $2qa \to \pi$.

c) Show that, in the limit $m = M$ the dispersion reduces to the one of monoatomic linear chain.